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NOMENCLATURE

English	
A	expression defined by Eq. (H.7)
$[A]$	matrices defined by Eqs. (4.4.10) and (5.6.16)
$A_{11,12,13,14}$	constants defined by Eqs. (4.2.16)
$A_{21,22,23,24}$	constants defined by Eqs. (4.2.16)
Ai	Airy function of the first kind
Bi	Airy function of the second kind
Bp	Benjamin's parameter
c	phase velocity of disturbance
$\bar{c}$	dimensionless phase velocity in Eq. (2.1.1)
c_0	speed of propagation of surface tension-gravity wave or Kelvin-Helmholtz wave
c_1	eigenvalue in liquid disturbance equations
c_2	eigenvalue in gas disturbance equations
c_p	specific heat at constant pressure
$\bar{c}_p$	gas/liquid specific heat ratio
C	expression defined by Eq. (H.8)
$\{C\}$	column matrix of constants of integration defined by Eqs. (4.4.11) and (5.6.17)
C_{1-8}	constants of integration in zero mass transfer problem
C_{1-14}	constants of integration in mass transfer problem
$d_{1,2}$	integrals defined by Eqs. (F.1) and (F.2)
D	diffusion coefficient, integral defined by Eq. (H.8), also expressions defined by Eqs. (3.3.6) and (4.2.16)
E	Euler number of gas

$E^{\pm}$	exponential defined in Eq. (4.3.12)
f	function in Eq. (G.5), also in Eq. (2.6.1)
F	Froude number of liquid
$F(x,y,t)$	function describing interface shape
$F_{1,2}$	functions defined by Eqs. (5.5.19) and (5.5.25)
F_k	factor defined by Eq. (6.1.5)
g	gravitation acceleration
$g_{1,2}$	functions described in Sec. 2.7(iv)
G	characteristic functions defined by Eqs. (4.4.11) and (5.6.17)
$G_{1,2}$	functions defined in Eqs. (D.10) and (D.14)
h	liquid layer depth
$h(t_1,\tau_1)$	integrand defined by Eq. (H.18)
$\overline{h}$	enthalpy
h_1	function defined by Eq. (5.4.10)
H	mass transfer function defined by Eq. (5.3.2)
H_1	function defined by Eq. (5.4.11)
I_{1-16}	integrals in zero mass transfer problem (Appendix E)
I_{1-48}	integrals in mass transfer problem (Appendix F)
$J_{1,2,3,4}$	integrals defined by Eqs. (5.5.17) and (5.5.18)
$\overline{J}_{1,2,3,4}$	integrals defined by Eqs. (5.5.14) and (5.5.15)
k	dimensionless wave number
$k_{1,2}$	thermal conductivity of liquid and gas respectively
$\overline{k}$	gas/liquid thermal conductivity ratio

K	constant in Eq. (3.3.1)
ℓ	latent heat of vaporization of liquid
Le	Lewis number
$\dot{m}$	steady-state mass transfer per unit time per unit area
m	parameter defined by Eq. (2.1.2)
n	exponent defined by Eq. (5.3.3), also number of zeros of Legendre polynomials
$\overline{n}$	unit normal to interface
N	expression defined by Eq. (3.3.6), also defined for summation (I.4)
p	static pressure
Pr	Prandtl number
P_2	expression used in Eq. (5.6.25)
q_i	a typical physical quantity
$Q_{1,2}$	expressions in Eqs. (5.6.26) and (5.6.27)
R	gas constant of vapor, also radius of curvature
$R_{1,2}$	Reynolds number of liquid and gas respectively
R_h	mass transfer Reynolds number in liquid
R_δ	mass transfer Reynolds number in gas
$\bar{\bar{R}}$	dimensionless group defined by Eq. (3.7.42)
s	small parameter in Eq. (2.5.1)
S	integral defined in Eq. (H.7)
S_1	function defined by Eq. (5.4.6)
$S_{1AA,2AA}$	integrals defined by Eq. (H.5)

$S_{1BA,2BA}$	integrals defined by Eq. (H.6)
t	time
t_1	transformation defined by Eq. (G.3)
$\tilde{t}$	dummy variable of integration
$T_{1,2}$	dimensional temperature of liquid and gas respectively
$\overline{T}$	ratio of interface temperature to external gas temperature
u	velocity component in x-direction
$\overline{u}$	ratio of interface velocity to gas velocity at the edge of the boundary layer
U	column matrices defined by Eqs. (4.4.14) and (5.6.20)
v	velocity component in y-direction
V	column matrices defined by Eqs. (4.4.10) and (5.6.16)
V_R	relative velocity vector
w	expression defined by Eq. (5.4.2)
W	Weber number, also Wronskian in Eq. (5.5.9)
x	co-ordinate in the plane of undisturbed interface of infinite extent
y	co-ordinate normal to the interface
$y_{1,2}$	homogeneous solutions used in method of variation of parameters
Y_2	expression used in Eq. (5.6.15)
$z_{1,2}$	arguments of Airy functions as defined in Eqs. (5.5.3) and (5.4.26)

Z	argument of Airy functions in Eqs. (4.3.7) - (4.3.10) and argument of characteristic function in Eq. (4.6.1)

<u>Greek</u>

α	dimensionless wave number
β	coefficient of volume expansion of liquid
γ	zeros of Legendre polynomials
Γ	surface tension
δ	boundary layer thickness
Δc_1	correction to c_1 in Newton-Raphson method
ε	ratio of liquid layer to boundary layer thickness
$\zeta_{1,2}$	transformations defined by Eqs. (4.3.4), (4.3.20), (5.4.8) and (5.4.18)
η	dimensionless vertical co-ordinate in gas
$\eta(x,t)$	function describing shape of the interface
θ	dimensionless temperature
κ	integral defined by Eq. (H.20)
λ	disturbance wavelength
Λ	dimensionless parameter defined by Eq. (3.7.41)
μ	coefficient of viscosity
$\overline{\mu}$	gas/liquid viscosity ratio
ν	kinematic viscosity
ξ	dimensionless vertical co-ordinate in liquid
ρ	density
$\overline{\rho}$	gas/liquid density ratio
σ	normal stress

τ	shear stress
$\tilde{\tau}$	dummy variable of integration
τ_1	transformation defined by Eq. (H.14)
ϕ	interface inclination (Fig. B.1)
Φ	viscous dissipation function
χ	vapor mass fraction
ψ	dimensionless vertical velocity component
ω	angular frequency of disturbance

Subscripts

1	liquid
2	gas or gas-vapor mixture
e	external inviscid gas flow
g	gas in mass transfer problem
i	imaginary part
if	interface
ℓam	laminar
r	real part
ref	reference condition
v	vapor in mass transfer problem
w	wall

Superscripts

~	dimensionless steady-state quantity

$\_$	dimensional unsteady quantity when subscripted, also dimensionless interface quantity when unsubscripted
$\hat{}$	dimensionless steady-state quantity
$'$	differentiation with respect to non-dimensional co-ordinate ξ
$\cdot$	differentiation with respect to non-dimensional co-ordinate η
$*$	turning point of Airy function

CHAPTER I

INTRODUCTION

1.1 Hydrodynamic Stability of a Gas/Liquid Interface

Hydrodynamic stability is one of the important branches of fluid mechanics. The original interest in this field was mainly in the area of transition of laminar to turbulent motion. In recent years, however, methods of hydrodynamic stability have been extended to treat the stability of the interface between two fluids. The motivation for studying dynamics of the interface comes from a variety of phenomena occurring in nature and in modern engineering problems. An example of the former which has fascinated the scientific mind for a long time is the generation of waves on the ocean surface by wind. Since the wind motion is practically incompressible, many of the earlier investigations were concerned with two incompressible fluids in parallel motion. With the advent of space age it was felt necessary to provide some means of cooling the spacecraft re-entering the atmosphere at hyper velocities. An efficient way of achieving this is to inject a liquid coolant through the nose of the spacecraft so that a layer of liquid is maintained on the windward surface. Thus, effective cooling can be accomplished if the liquid film stays adhered to the surface. Therefore stability of liquid films in high speed gas environments received a great deal of attention in the 1970's.

It is a fact of experience that wind blowing over water generates waves on the interface and that, under certain conditions, these waves grow in size. Once a wave reaches a significant height droplets are stripped off from the wave crests and get entrained in the air. Hence the liquid is lost not only due to the simple process of evaporation but also due to the entrainment. In fact, current investigations show that the latter mechanism is more dominant.

It is the purpose of the present work to introduce the element of interface mass transfer into the stability problem. Several simplifying assumptions have been made to gain a first insight into the nature of the problem and to permit analytical tractability. To this end only the linear stability problem is considered. Physically, this assumption implies that the waves on the interface are of infinitesimal amplitude or, put differently, the wave amplitude is much smaller than the wavelength. It is unlikely that the mechanism of entrainment would be significant under these circumstances and hence only the mass transfer due to evaporation is taken into account. Another key assumption introduced in the present study is that the gas motion is incompressible. It may be argued that mass transfer effects would be negligible in the incompressible case and therefore it is the case of compressible gas that is worth examining. However, in many natural processes and engineering problems the air flow is incompressible; for example, interactions between wind and ocean surface, dispersion of pollutants into the atmosphere from large bodies of water, two-phase flows and industrial drying processes. Apart from these practical applications, the fact that the incompressible problem has not yet been completely understood provides motivation for the present work. Also, the experience gained in the study of boundary layer stability has shown that the compressible problem is considerably more complicated than its incompressible counterpart. Indeed, a complete spectrum of eigenvalues of the simple Blasius flat plate boundary layer was obtained only very recently. Thus an examination of the incompressible case appears to be the logical first step. Under the assumption of incompressible gas and the restriction of evaporative mass transfer the rate of mass injection into the gas is expected to be small. It will be shown that under these conditions the exponential steady-state velocity, temperature and concentration profiles reduce to linear profiles. In addition to the assumptions outlined above the classical assumption of parallel mean (or steady-state) flow is also made. It is then possible to solve the governing equations in a closed form.

The present work is an attempt to determine a qualitative and quantitative estimate of the influences of mass transfer on the liquid film stability problem. This endeavor is pursued with the intention of producing a useful analytical framework which can treat such interesting aspects as (i) amplification and phase velocity curves (ii) neutral stability curves (iii) stress, temperature and concentration perturbations at the interface and (iv) energy transfer mechanisms. The analysis presented here departs from the customary assumption of neglecting the instabilities in gas, which amounts to ignoring the eigenvalue in gas disturbance equations. The consequences of relaxing this assumption are carefully examined.

1.2 Review of Literature

It has been recognized for a long time that waves are initiated on a liquid surface due to some kind of instability. The growth or decay of these waves depends on whether energy is transferred to or removed from them. Hence the study of mechanisms of energy transfer has received considerable attention in the literature. Ursell[1] provides an excellent survey of wind generated waves on deep water. A widely respected theory of wave initiation is due to Miles[2,3]. He proved that the rate of energy transfer to a wave of speed c is proportional to the profile curvature $-U''(c)$ in the gas. This theory predicts exponential wave growth whereas experiential observations show that the initial wave growth is linear. It was Phillips[4] who explained the initial linear wave growth by proposing a resonance mechanism between turbulent pressure fluctuations and interface disturbances. A combined Miles-Phillips[5] theory provides a very good qualitative description of wave initiation and growth. Lighthill[6] has offered a remarkable physical interpretation of Miles' theory.

There are three principal types of instabilities which are of interest in interface stability problems. These are discussed below and the relevant literature is reviewed.

(i) Tollmien-Schlichting Instability:

The Tollmien-Schlichting mechanism has been studied extensively in connection with the stability of laminar boundary layers (Ref. 7).

This instability is the result of continual energy transfer from the mean flow to the disturbance. Benjamin[8,9], Landahl[10] and Skripachev[11] investigated the stability of laminar boundary layers over flexible surfaces. Their analyses predicted three different stability modes, viz. modified forms of the Tollmien-Schlichting mode, gravity waves (body force instabilities) and Kelvin-Helmholtz instability. The latter two types will be described shortly. In recent years the higher stability modes associated with the incompressible, laminar flat plate boundary layer have been obtained by Gastor and Jordinson[12] and Mack[13]. Mack's calculations show that the number of eigenvalues is finite for a velocity profile which is sufficiently smooth at the outer edge of the boundary layer. This work, along with the analysis of DiPrima and Habetler[14], shows that in a finite interval there exists an infinite discrete eigenvalue spectrum. Gallagher and Mercer[15,16] and Deardorff[17] examined the eigenvalue spectrum of a plane Couette flow and concluded that it is stable to infinitesimal disturbances. The stability of a plane Couette-Poiseuille flow with uniform cross flow has been investigated by Haines[18].

(ii) Instability due to shear and pressure perturbations at the interface:

Small wavy disturbances on the interface become unstable when pressure and shear perturbations overcome the stabilizing effect of gravity and surface tension. A special case of this instability mechanism is the well-known Kelvin-Helmholtz instability (Ref. 19) between two incompressible, inviscid fluids in parallel motion. Miles[20] generalized this simple case to parallel shear flows. In a classic paper Benjamin[21] presented results for pressure and shear stress perturbations exerted by a viscous incompressible fluid on a wavy wall. He considered the cases of rigid, flexible yet solid, and a completely mobile wavy wall. In the course of his investigation he also laid down the requirements under which a moving wavy interface could be approximated as rigid. These results were applied by Craik[22] to wind-generated waves in thin liquid films. He discovered that the dominant instability mechanism in thin films is due to shear perturbations. Lighthill[23] analyzed the shear and pressure perturbations exerted by a viscous compressible gas on a rigid wavy wall neglecting the effects of heat and mass transfer at the wall. This work was amplified by Inger[24] to include heat and mass transfer effects.

(iii) Rayleigh-Taylor instability:

This instability is due to body forces pointing from the heavier fluid to lighter fluid (Ref. 25). The problem of liquid film stability in a body force field was solved by Nayfeh and Saric[26].

Several investigators have studied the linear stability of liquid-gas interface under various conditions and a concise summary of their works is provided in Table I at the end of this chapter. It should be noted that the liquid motion is treated as incompressible and laminar in all cases, and effects of mass transfer are neglected unless mentioned otherwise. Major features of the present work are also listed in this table to facilitate comparison with other investigations.

1.3 Outline of Present Work

It was pointed out in the previous section that steady-state exponential mass transfer profiles reduce to linear profiles without mass transfer when the evaporation rate is small. This suggests that the zero mass transfer problem with linear profiles be solved first. Since the zero mass transfer problem is relatively simpler, the experience gained in its formulation would be very useful. Also, this problem can be used to evaluate the importance of the often-made assumption of neglecting gas instability. The formulation of the zero mass transfer problem is described in Chapter II and its solution is presented in Chapter IV. Two methods of solution have been used - (i) a small perturbation scheme which yields one eigenvalue in the long wavelength approximation and (ii) exact analytical solution for arbitrary wavelengths.

The mass transfer formulation is considered in Chapter III and follows closely the procedures of Chapter II. Chapter V is concerned with the solution of the mass transfer problem. All the important steps of mathematical analysis are included in Chapters II through V and the details are confined to the Appendices. This was done in order to preserve clarity of the presentation. The results of numerical computations are given in the form of amplification and phase velocity curves in Chapter VI. The conclusions derived from the present investigation are summarized in Chapter VII.

TABLE I

Investigator	Ref. No.	Gas Characteristics	Liquid Characteristics	Remarks
Lock (1953)	27	Incompressible, laminar, parallel flow	Parallel flow, infinite depth	Restricted investigation to critical point inside the gas. Solved Orr-Sommerfeld equation using Tietjen's function. Found two distinct modes of instability (i) Air waves which correspond to modified Tollmien-Schlichting instability and (ii) water waves which are disturbances propagating with approximately the speed of gravity-surface tension waves.
Feldman (1957)	28	Incompressible, laminar, linear velocity profile in the boundary layer	Parallel flow, finite depth, linear velocity profile	Considered critical point inside the liquid and analyzed stability for $\alpha R_{liq} >> 1$. Obtained the apparently erroneous result that both gravity and surface tension are destabilizing.
Miles (1960)	29	Gas assumed absent	Parallel flow, finite depth, linear velocity profile, free surface	Studied critical point inside the liquid, derived asymptotic solution for $R_{liq} \to \infty$. Established a minimum critical Reynolds number (based on liquid depth) $R_{liq} = 203$ for the onset of instability.

Investigator	Ref. No.	Gas Characteristics	Liquid Characteristics	Remarks
Cohen & Hanratty (1965)	30	Incompressible, turbulent, channel flow	Used 'thick' water films in experiments, assumed linear velocity profile in analysis	Performed experiments on relatively thick liquid films. Observed waves travelling at faster than interface speeds, i.e. disturbances with critical points inside the gas. Analysis limited to large values of R_{liq}. Major conclusion was that the dominant mechanism of energy transfer is due to the work done by pressure perturbation in phase with the wave slope and shear perturbation in phase with the wave height.
Craik (1966)	22	Incompressible, turbulent, channel flow	Used 'thin' water films in experiments, assumed linear velocity profile in analysis	Investigated experimentally very thin liquid films. Observed disturbances travelling at speeds slower than the interface speed. Limited analysis to $R_{liq} \to 0$. Discovered the mechanism of shear perturbation instability wherein the energy transfer is due to the work done by shear perturbation in phase with the wave slope and pressure perturbation in phase with the wave height.

Investigator	Ref. No.	Gas Characteristics	Liquid Characteristics	Remarks
Plate, et. al. (1968)	31	Incompressible, turbulent	'thick' water films used in experiment	Measured accurately very small surface undulations. Concluded that the mechanism of turbulence proposed by Phillips[4] is not responsible for initiation of waves. Confirmed validity of Miles's[5] theory with modifications for moving interface.
Chang & Russell (1965)	32	Compressible, inviscid, parallel and uniform	Inviscid, parallel, infinite depth	Found that liquid adjacent to a supersonic stream is more unstable than liquid adjacent to a subsonic stream.
Nachtsheim (1970)	33	Compressible, inviscid, parallel and uniform	Parallel flow, linear velocity profile, finite depth	Analyzed stability w.r.t. 3D disturbances. Examined only those modes with critical point inside the gas. Demonstrated that the mechanism of wave generation is supersonic wave drag. Obtained solutions for $R_{liq} \rightarrow 0$.

Investigator	Ref. No.	Gas Characteristics	Liquid Characteristics	Remarks
Saric & Marshall (1971)	34	Supersonic air stream with $M \geq 4.95$ was used in experiments	Water, glycerin and their mixtures in various proportions	Experimented with injection over blunt wedges and cones. Observed disturbances both with critical points inside gas and liquid. Their 'slow' wave results (i.e. critical point inside the liquid) were in order of magnitude agreement with Miles's[2] theory.
Starkenberg (1972)	35	Compressible, inviscid, parallel and uniform	Parallel flow, linear velocity profile	Repeated Nachtscheim's (Ref. 33) work for 2D disturbances and moderate values of R_{liq}.
Creski & Starkenberg (1973)	36	Supersonic air stream with $M = 8$	Parallel flow, linear velocity profile	Carried out experimental study of liquid coolant injected in the nose region of a blunt slender body. Gave steady-state analysis for effects of vaporization and mass entrainment. This analysis is in gross error when compared with experimental data. Their unsteady analysis does not include mass transfer effects.

Investigator	Ref. No.	Gas Characteristics	Liquid Characteristics	Remarks
Bordner et al. (1973)	37	Compressible, viscous, parallel flow, Prandtl number unity, constant properties, zero pressure gradient	Parallel flow, linear velocity profile	Inger's (Ref. 24) model is used to calculate shear and pressure perturbations exerted by the compressible gas on the interface. Analysis is valid for all wave numbers and R_{liq}. Neglected eigenvalue in gas disturbance equations by assuming steady interface.
Nayfeh & Saric (1971)	26	Incompressible, or compressible, inviscid, parallel and uniform	Parallel flow, parabolic velocity profile	Investigated body force effects on stability. Obtained solutions for long wavelength disturbances. Employed Craik's[22] shear and pressure perturbation expressions.
Gater & L'Ecuyer (1970)	38	Incompressible, turbulent at temperatures of 40-60°F and pressures up to 150 psia	'thick' (0.03") films of methanol, water, butanol and RP-1	Experiments show that mass transfer due to entrainment is several times greater than that due to evaporation. Obtained empirical expressions for entrainment mass transfer.

Investigator	Ref. No.	Gas Characteristics	Liquid Characteristics	Remarks
Kotake (1973)	39	Incompressible, laminar, parallel flow, arbitrary Prandtl & Schmidt numbers, constant properties and zero pressure gradient	Parallel flow, infinite depth	Analyzed the steady-state problem and found that interface evaporation leads to decrease in skin friction and heat transfer coefficients.
Kotake (1974)	40	Incompressible, laminar, parallel flow, linear velocity, temperature and concentration profiles. Constant properties, zero pressure gradient, Prandtl & Schmidt numbers equal to unity	Parallel flow, infinite depth	Obtained shear, pressure, temperature and concentration perturbation expressions for a rigid wavy wall. Found skin friction to be more sensitive to the phase-changing interface than heat transfer.

Investigator	Ref. No.	Gas Characteristics	Liquid Characteristics	Remarks
Kotake (1974)	41	Incompressible, laminar, parallel flow, constant properties, zero pressure gradient.	Parallel flow, infinite depth	Obtained neutral stability curves with evaporative mass transferred and compared with Lock's (Ref. 27) work without mass transfer. However, only the part of Lock's curves which corresponds to air waves is presented. The author does not mention any other stability mode. In this sense the work is incomplete.
Joshi & Schetz (1976)		Incompressible, laminar, parallel flow, linear velocity, temperature and concentration profiles, constant properties, zero pressue gradient, Prandtl and Schmidt numbers equal to unity	Parallel flow, linear velocity profile, finite depth	Formulated the unsteady or disturbance problem by including instability in both gas and liquid. Obtained several modes (eigenvalues) for Craik's (Ref. 22) experimental data. Results show that neglecting gas instability would predict a stable interface at moderate values of wave numbers, whereas inclusion of gas instability would predict an unstable interface. Mass transfer computations show that evaporation tends to destabilize the interface at moderate values of wave numbers but does not affect the neutral stability point of this particular mode.

CHAPTER II

FORMULATION OF THE ZERO MASS TRANSFER PROBLEM

2.1 The Purpose of Solving the Zero Mass Transfer Problem

1. It was mentioned in Sec. 1.2 that different types of mechanisms have been proposed to explain the transfer of energy to the surface waves. These mechanisms are briefly summarized below.

(i) Energy transfer from the mean velocity profile in the gas:

This mechanism was proposed by Miles[2] and Benjamin[21]. The critical point (y-location at which the disturbance phase speed equals the u-velocity component) of the velocity profile lies within the gas boundary layer. Energy is fed from the mean flow to the interface disturbance continually with time resulting in a Tollmien-Schlichting type of instability. However, the experiments of Cohen and Hanratty[30] and Plate, et al.[31] indicate otherwise.

(ii) Energy transfer from the mean velocity profile in the liquid:

The critical point lies inside the liquid layer and again the energy transfer to the interface occurs as a result of Tollmien-Schlichting instability. This mechanism was investigated by Feldman and Miles[29] for large liquid Reynolds numbers. In this mode the phase speed c_r is less than the interface velocity ($c_r < u_{if}$) and these waves are sometimes referred to as 'slow waves.' These waves have been observed in the experiments of Craik[22] and Saric and Marshall[34]. However, Cohen and Hanratty reject this mode of energy transfer because they did not observe slow waves in their experiments. One reason may be that they did not use sufficiently small thicknesses in their work. Saric and Marshall[34] mention that the occurrence of slow waves may be due to nonlinear effects.

(iii) Energy transfer due to pressure and shear perturbation exerted by gas on the disturbed interface:

The classical Kelvin-Helmholtz stability is a special case of this mechanism. Cohen and Hanratty[30] proposed that this is the sole mechanism of energy transfer. They found that for 'fast' waves (critical point inside the gas or $c_r > u_{if}$) or 'thick' films the component of pressure perturbation in phase with the wave elevation are dominant. Craik[22], however, discovered that for 'slow' waves on very thin films, the pressure perturbation component in phase with the wave elevation and shear perturbation component in phase with the wave slope, are dominant.

Another model based on energy transfer due to pressure variations at the interface is Jeffrey's[1] sheltering hypothesis. It is improbable that the mechanism of sheltering (i.e. drag force exerted on a wave due to flow separation near the wave crest) will be important in the case of waves with amplitudes very small compared to their wavelengths. The latter is assumed in all the linear analyses including the present work.

(iv) Energy transfer due to a resonance mechanism between turbulent pressure fluctuations and surface disturbances:

Proposed by Phillips[4], this mechanism is believed to be responsible for initiation of short waves on the interface. Cohen and Hanratty[30] rule out this mode on the basis that a smooth liquid surface is 'observed' even in the presence of turbulent flow (not a very convincing argument at all!). Plate, et al.[31] obtained turbulent pressure fluctuations at the liquid surface indirectly through the measurement of longitudinal velocity fluctuation u'. Their experiments come closest to verifying Phillips' theory. They conclude that the resonance mechanism does not appear to be significant and this may be a justification for neglecting turbulence interactions in the present work. One of the aims of solving the zero mass transfer problem is to try to shed some light on the mechanisms of energy transfer.

2. As seen earlier in Sec. 1.2 some investigators have a priori neglected the instability in the gas. For instance, Nachtscheim[33] and Starkenberg[35] observe that $c/u_e << 1$ for inviscid supersonic external flow, where c is the phase velocity of the disturbance and u_e is the gas velocity. Bordner[37] presents an order of magnitude argument for neglecting c in the gas disturbance equations when the external flow is viscous and compressible. In his work on cross-hatching Inger[24] also assumes that the interface behaves like a rigid wavy wall relative to the gas flow. Craik[22] analysed thin liquid films using the pressure and shear perturbations derived by Benjamin[21] for a rigid wavy boundary. This amounts to having the critical point located at the wavy boundary and it also means that the phase speed c is negligible in the gas disturbance equations. It ought to be mentioned at this point that the assumptions (a) the phase speed relative to gas speed is negligible (b) the interface is steady and rigid for gas disturbance equations and (c) the critical point in the

gas is located at the wavy boundary, are all equivalent.

Craik[22] indicates that even for a viscous incompressible gas $c/u_e \ll 1$ is sufficient to make the above assumption (which is understandable in the case of an inviscid compressible gas). It is questionable how Craik's assumption (i.e. $c/u_e \ll 1$) satisfies the requirement laid down by Benjamin[21] that

$$\frac{m\bar{c}}{u'(0)} \ll 1 \tag{2.1.1}$$

in order to make the rigid wavy wall assumption. In the last inequality $\bar{c} = c/u_e$, $u'(0)$ is the non-dimensional slope of the velocity profile evaluated at wall and

$$m = [\alpha R u'(0)]^{1/3} \tag{2.1.2}$$

where R is the gas Reynolds number based on some characteristic thickness and $\alpha = 2\pi/\lambda$, λ being the wavelength of the rigid wavy wall. Thus Benjamin's criterion requires that

$$[\alpha R u'(0)]^{1/3} \frac{c/u_e}{u'(0)} \ll 1 \tag{2.1.3}$$

It is clear from Eq. (2.1.3) that for a low speed (incompressible) gas with moderate Reynolds number and sufficiently large α(small ripples on the interface) Benjamin's criterion may not hold. Hence c/u_e may be very small compared to unity but still Eq. (2.1.1) could be violated and consequently, a rigid wavy interface assumption cannot be made. An illustrative example will be considered in Chapter VI.

In the present analysis the external gas is viscous and incompressible and hence the rigid/steady interface assumption is not made. Hence phase speed terms appear in both the gas and liquid disturbance equations. The stability problem is solved both with and without this assumption, thus providing a method of checking its validity.

3. An incompressible, viscous, laminar flow of gas with constant properties over an incompressible laminar viscous liquid is considered. A turbulent velocity profile in the gas, however, can be treated as a 'quasi-laminar' profile with augmented viscosity. The assumption of laminar flow simplifies the analysis greatly but it is not very realistic. In the turbulent case it would imply that the laminar sublayer be large compared to the disturbance wavelength -- a requirement rarely met in practice.

The zero mass transfer problem is solved for a linear steady state velocity profile in both gas and liquid. The linear profile in the gas can be justified to some extent in the laminar case but it may be a poor approximation in the turbulent case. The laminar flow and linear velocity profile assumption can be justified for the liquid since the liquid Reynolds number is usually small and since Craik's experiment confirms a linear profile. The linear velocity problem is solved in the present work due to the following reasons:

(i) the Orr-Sommerfeld equation has an exact solution.

(ii) the linear velocity profile is the simplest viscous profile which represents a physically possible flow.

(iii) the linear stability of plane Couette flow has been extensively studied (e.g. Gallagher and Mercer[15], Deardorff[17]) and it has been found to be unconditionally stable.

(iv) The exponential steady-state velocity profiles with mass transfer reduce to linear profiles for small (but non-zero) rates of mass transfer. This last reason, in particular, provided motivation for studying the zero mass transfer problem with linear steady-state profiles.

2.2 The Steady-State Problem

As mentioned in Sec. 1.1 the steady-state or the mean flow is assumed to be incompressible and parallel (i.e. $\frac{\partial}{\partial x} \equiv 0$ and $v_1 = v_2 \equiv 0$). The liquid motion is assumed to be a plane Couette flow and hence it is an exact solution of the full Navier-Stokes equations. The gas motion, on the other hand, is assumed to be a boundary layer flow which is approximately parallel.

Let h and δ be the height of the liquid Couette flow and the boundary layer thickness respectively (Fig. 1a). The gas Prandtl number Pr_2 is assumed to be unity so that the thermal and velocity boundary layer thicknesses are identical. Let subscripts 1 and 2 denote the liquid and gas respectively. The steady-state governing equations are --

Liquid:

1. x-momentum

$$\frac{d^2\tilde{u}_1}{dy^2} = 0 \qquad (2.2.1)$$

2. y-momentum

$$\frac{d\tilde{p}_1}{dy} = -\rho_1 g \qquad (2.2.2)$$

3. Energy

$$\frac{d^2\tilde{T}_1}{dy^2} = 0 \qquad (2.2.3)$$

Gas:

4. x-momentum

$$\frac{d^2\tilde{u}_2}{dy^2} = 0 \qquad (2.2.4)$$

5. y-momentum

$$\frac{d\tilde{p}_2}{dy} \simeq 0 \qquad (2.2.5)$$

6. Energy

$$k_2 \frac{d^2\tilde{T}_2}{dy^2} = -\mu_2\left(\frac{d\tilde{u}_2}{dy}\right)^2 \simeq 0 \qquad (2.2.6)$$

These equations show that all the flow quantities vary only with respect to the vertical co-ordinate y. In Eq. (2.2.6) the viscous dissipation term is neglected consistent with the assumption of incompressibility. It may also be mentioned here that continuity equations for the gas and liquid are identically satisfied due to the parallel flow assumption. Eqs. (2.2.1) - (2.2.6) are six equations in six unknowns $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{T}_1$, $\tilde{T}_2$. The combined order of this system is 10 and an equal number of boundary conditions is required for a unique solution. These conditions are --

1. No slip at the wall

$$\tilde{u}_1(-h) = 0 \tag{2.2.7}$$

2. Boundary layer edge condition on velocity

$$\tilde{u}_2(\delta) = u_e \tag{2.2.8}$$

3. No slip in tangential velocities at the interface

$$\tilde{u}_1(0) = \tilde{u}_2(0) \tag{2.2.9}$$

4. Balance of shear stresses at the interface

$$\mu_1 \frac{d\tilde{u}_1}{dy}\bigg|_{y=0} = \mu_2 \frac{d\tilde{u}_2}{dy}\bigg|_{y=0} \tag{2.2.10}$$

5. Constant temperature or adiabatic wall

$$\tilde{T}_1(-h) = T_w \quad \text{or} \quad \frac{d\tilde{T}_1}{dy}\bigg|_{y=-h} = 0 \tag{2.2.11}$$

6. Boundary layer edge condition on temperature

$$\tilde{T}_2(\delta) = T_e \tag{2.2.12}$$

7. Energy balance at the interface

$$k_1 \frac{d\tilde{T}_1}{dy}\bigg|_{y=0} = k_2 \frac{d\tilde{T}_2}{dy}\bigg|_{y=0} \tag{2.2.13}$$

8. No jump in temperature at the interface

$$\tilde{T}_1(0) = \tilde{T}_2(0) \tag{2.2.14}$$

9. Boundary layer edge condition on pressure

$$\tilde{p}_2(\delta) = p_e \tag{2.2.15}$$

10. Balance of normal stresses at the interface

$$\tilde{p}_1(0) = \tilde{p}_2(0) \tag{2.2.16}$$

The solution of Eqs. (2.2.1) - (2.2.6) subject to (2.2.7) - (2.2.16) is

Liquid velocity profile

$$\frac{\tilde{u}_1}{u_e} = \frac{\mu_2}{\mu_1}\frac{h}{\delta}\,\frac{1+\frac{y}{h}}{1+\frac{\mu_2}{\mu_1}\frac{h}{\delta}} \tag{2.2.17}$$

Gas velocity profile

$$\frac{\tilde{u}_2}{u_e} = \frac{\frac{\mu_2}{\mu_1}\frac{h}{\delta}+\frac{y}{\delta}}{1+\frac{\mu_2}{\mu_1}\frac{h}{\delta}} \tag{2.2.18}$$

Liquid pressure profile

$$\tilde{p}_1 = p_e - \rho_1 g y \tag{2.2.19}$$

Gas pressure profile

$$\tilde{p}_2 = \tilde{p}_e \tag{2.2.20}$$

Liquid temperature profile

$$\frac{\tilde{T}_1}{T_e} = \frac{k_2}{k_1}\frac{h}{\delta}\,\frac{1 - \frac{T_w}{T_e}}{1 + \frac{k_2}{k_1}\frac{h}{\delta}}\,\frac{y}{h} + \frac{\frac{T_w}{T_e} + \frac{k_2}{k_1}\frac{h}{\delta}}{1 + \frac{k_2}{k_1}\frac{h}{\delta}} \quad \text{constant temperature wall} \tag{2.2.21}$$

$$= 1 \quad \text{adiabatic wall} \tag{2.2.22}$$

Gas temperature profile

$$\frac{\tilde{T}_2}{T_e} = \frac{1 - \frac{T_w}{T_e}}{1 + \frac{k_2}{k_1}\frac{h}{\delta}}\,\frac{y}{\delta} + \frac{\frac{T_w}{T_e} + \frac{k_2}{k_1}\frac{h}{\delta}}{1 + \frac{k_2}{k_1}\frac{h}{\delta}} \quad \text{constant temperature wall} \tag{2.2.23}$$

$$= 1 \quad \text{adiabatic wall} \tag{2.2.24}$$

Interface quantities are obtained from Eqs. (2.2.17) - (2.2.24) by putting $y = 0$. Thus

Interface velocity

$$\frac{u_{if}}{u_e} = \frac{\frac{\mu_2}{\mu_1}\frac{h}{\delta}}{1 + \frac{\mu_2}{\mu_1}\frac{h}{\delta}} \tag{2.2.25}$$

Interface pressure

$$\frac{p_{if}}{p_e} = 1 \tag{2.2.26}$$

Interface temperature

$$\frac{T_{if}}{T_e} = \frac{\dfrac{T_w}{T_e} + \dfrac{k_2}{k_1}\dfrac{h}{\delta}}{1 + \dfrac{k_2}{k_1}\dfrac{h}{-}} \quad \text{constant temperature wall} \tag{2.2.27}$$

$$= 1 \quad \text{adiabatic wall} \tag{2.2.28}$$

2.3 The Unsteady Problem -- Governing Equations

When the steady interface configuration of Sec. 2.2 is disturbed the resulting unsteady, two dimensional, incompressible motion is governed by the following equations:

Liquid:

1. Continuity

$$\frac{\partial \bar{u}_1}{\partial x} + \frac{\partial \bar{v}_1}{\partial y} = 0 \tag{2.3.1}$$

2. x-momentum

$$\frac{\partial \bar{u}_1}{\partial t} + \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x} + \bar{v}_1 \frac{\partial \bar{u}_1}{\partial y} = -\frac{1}{\rho_1}\frac{\partial \bar{p}_1}{\partial x} + \nu_1\left(\frac{\partial^2 \bar{u}_1}{\partial x^2} + \frac{\partial^2 \bar{u}_1}{\partial y^2}\right) \tag{2.3.2}$$

3. y-momentum

$$\frac{\partial \bar{v}_1}{\partial t} + \bar{u}_1 \frac{\partial \bar{v}_1}{\partial x} + \bar{v}_1 \frac{\partial \bar{v}_1}{\partial y} = -\frac{1}{\rho_1}\frac{\partial \bar{p}_1}{\partial y} + \nu_1\left(\frac{\partial^2 \bar{v}_1}{\partial x^2} + \frac{\partial^2 \bar{v}_1}{\partial y^2}\right) \tag{2.3.3}$$

4. Energy (static enthalpy form)

$$\rho_1 \left(\frac{\partial \bar{h}_1}{\partial t} + \bar{u}_1 \frac{\partial \bar{h}_1}{\partial x} + \bar{v}_1 \frac{\partial \bar{h}_1}{\partial y} \right) = \frac{\partial \bar{p}_1}{\partial t} + \bar{u}_1 \frac{\partial \bar{p}_1}{\partial x} + \bar{v}_1 \frac{\partial \bar{p}_1}{\partial y}$$

$$+ k_1 \left(\frac{\partial^2 \bar{T}_1}{\partial x^2} + \frac{\partial^2 \bar{T}_1}{\partial y^2} \right) + \mu_1 \bar{\Phi}_1 \qquad (2.3.4)'$$

Combining this equation with the equation of state for the liquid

$$\bar{h}_1 = \bar{h}_1 \ (\bar{p}_1, \ \bar{T}_1)$$

and neglecting viscous dissipation and pressure gradient terms it can be shown that (Appendix A)

$$\rho_1 c_{p1} \left(\frac{\partial \bar{T}_1}{\partial t} + \bar{u}_1 \frac{\partial \bar{T}_1}{\partial x} + \bar{v}_1 \frac{\partial \bar{T}_1}{\partial y} \right) = k_1 \left(\frac{\partial^2 \bar{T}_1}{\partial x^2} + \frac{\partial^2 \bar{T}_1}{\partial y^2} \right) \qquad (2.3.4)$$

Gas:

5. Continuity

$$\frac{\partial \bar{u}_2}{\partial x} + \frac{\partial \bar{v}_2}{\partial y} = 0 \qquad (2.3.5)$$

6. x-momentum

$$\frac{\partial \bar{u}_2}{\partial t} + \bar{u}_2 \frac{\partial \bar{u}_2}{\partial x} + \bar{v}_2 \frac{\partial \bar{u}_2}{\partial y} = - \frac{1}{\rho_2} \frac{\partial \bar{p}_2}{\partial x} + \nu_2 \left(\frac{\partial^2 \bar{u}_2}{\partial x^2} + \frac{\partial^2 \bar{u}_2}{\partial y^2} \right) \qquad (2.3.6)$$

7. y-momentum

$$\frac{\partial \bar{v}_2}{\partial t} + \bar{u}_2 \frac{\partial \bar{v}_2}{\partial x} + \bar{v}_2 \frac{\partial \bar{v}_2}{\partial y} = -\frac{1}{\rho_2}\frac{\partial \bar{p}_2}{\partial y} + \nu_2\left(\frac{\partial^2 \bar{v}_2}{\partial x^2} + \frac{\partial^2 \bar{v}_2}{\partial y^2}\right) \tag{2.3.7}$$

8. Energy (static enthalpy form)

$$\rho_2\left(\frac{\partial \bar{h}_2}{\partial t} + \bar{u}_2 \frac{\partial \bar{h}_2}{\partial x} + \bar{v}_2 \frac{\partial \bar{h}_2}{\partial y}\right) = \frac{\partial \bar{p}_2}{\partial t} + \bar{u}_2 \frac{\partial \bar{p}_2}{\partial x} + v_2 \frac{\partial \bar{p}_2}{\partial y}$$

$$+ k_2\left(\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2}\right) + \mu_2 \bar{\Phi}_2 \tag{2.3.8)'$$

combining this equation with the relation for an ideal gas

$$(\partial h/\partial T)_p = c_p$$

and neglecting pressure gradient and viscous dissipation terms the result is

$$\rho_2 c_{p2}\left(\frac{\partial \bar{T}_2}{\partial t} + \bar{u}_2 \frac{\partial \bar{T}_2}{\partial x} + \bar{v}_2 \frac{\partial \bar{T}_2}{\partial y}\right) = k_2\left(\frac{\partial^2 \bar{T}_2}{\partial x^2} + \frac{\partial^2 \bar{T}_2}{\partial y^2}\right) \tag{2.3.8}$$

It is assumed that the unsteady motion is confined to the region $-h \leq y \leq \delta$, i.e. the steady-state gas boundary layer thickness δ is unaltered. This is justifiable in view of the assumption of small perturbation motion to be introduced later (Sec. 2.5).

2.4 The Unsteady Problem -- Boundary Conditions

Eqs. (2.3.1) - (2.3.8) form a system of elliptic partial differential equations requiring boundary conditions to be specified along the entire boundary of the domain in (x,y) plane. However, it will be apparent in Sec. 2.6 that the unsteady perturbations are sinusoidal with respect to the x co-ordinate and hence bounded. Thus boundary conditions need be specified only along the y co-ordinate direction and along the interface. In this section the appropriate boundary and interface conditions are developed.

1. No slip at the wall

$$\bar{u}_1(x,y,t) = 0 \quad , \; y = -h \tag{2.4.1}$$

2. Boundary layer edge condition on the u-velocity component

$$\bar{u}_2(x,y,t) = u_e \quad , \; y = \delta \tag{2.4.2}$$

3. No slip in the tangential velocity at the interface (Fig. 2)

Let $\bar{V}_{R_1}$ and $\bar{V}_{R_2}$ be the liquid and gas velocity vectors at point p relative to the interface. The condition of no slip requires that the components of $\bar{V}_{R_1}$ and $\bar{V}_{R_2}$ along the interface be continuous, i.e.

$$\bar{V}_{R_1} \cdot d\bar{s} = \bar{V}_{R_2} \cdot d\bar{s}$$

where $d\bar{s}$ is a directed line segment of the interface at point p. Now,

$$\bar{V}_{R_1} = \bar{V}_1 - \bar{V}_{if}$$

and

$$\bar{V}_{R_2} = \bar{V}_2 - \bar{V}_{if}$$

where $\bar{V}_1$ and $\bar{V}_2$ are liquid and gas velocity vectors respectively relative to a stationary observer. $\bar{V}_{if}$ is the interface velocity vector with respect to a stationary observer. Hence,

$$(\bar{V}_1 - \bar{V}_{if}) \cdot d\bar{s} = (\bar{V}_2 - \bar{V}_{if}) \cdot d\bar{s}$$

or

$$\bar{V}_1 \cdot d\bar{s} = \bar{V}_2 \cdot d\bar{s}$$

Since

$$\bar{V}_1 = (\bar{u}_1, \bar{v}_1) \ , \ \bar{V}_2 = (\bar{u}_2, \bar{v}_2) \text{ and } d\bar{s} = (dx, dy)$$

$$\bar{u}_1 dx + \bar{v}_1 dy = \bar{u}_2 dx + \bar{v}_2 dy$$

i.e.

$$(\bar{u}_2 - \bar{u}_1) = (\bar{v}_1 - \bar{v}_2) \frac{dy}{dx}$$

If $y = \eta(x,t)$ is the equation describing the unsteady interface, the above relation gives

$$\bar{u}_2 - \bar{u}_1 = (\bar{v}_1 - \bar{v}_2) \frac{\partial \eta}{\partial x} \quad \text{at } y = \eta(x,t) \qquad (2.4.3)$$

For a flat interface $\eta_x = 0$ and (2.4.3) reduces to (2.2.9).

4. Balance of shear stresses at the interface

$$\tau_1 = \tau_2 \text{ on } y = \eta(x,t)$$

It can be shown by considering the equilibrium of a triangular element at the interface (Appendix B) that

$$\tau = \frac{1-\eta_x^{\,2}}{1+\eta_x^{\,2}}\,\mu\left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right) - \frac{2\mu\eta_x}{1+\eta_x^{\,2}}\left(\frac{\partial\bar{u}}{\partial x} - \frac{\partial\bar{v}}{\partial y}\right)$$

Thus shear balance requires that

$$\frac{1-\eta_x^{\,2}}{1+\eta_x^{\,2}}\,\mu_2\left(\frac{\partial\bar{u}_2}{\partial y} + \frac{\partial\bar{v}_2}{\partial x}\right) - \frac{2\mu_2\eta_x}{1+\eta_x^{\,2}}\left(\frac{\partial\bar{u}_2}{\partial x} - \frac{\partial\bar{v}_2}{\partial y}\right)$$

$$= \frac{1-\eta_x^{\,2}}{1+\eta_x^{\,2}}\,\mu_1\left(\frac{\partial\bar{u}_1}{\partial y} + \frac{\partial\bar{v}_1}{\partial x}\right) - \frac{2\mu_1\eta_x}{1+\eta_x^{\,2}}\left(\frac{\partial\bar{u}_1}{\partial x} - \frac{\partial\bar{v}_1}{\partial y}\right)$$

$$\text{on } y = \eta(x,t) \tag{2.4.4}$$

For a flat interface $\eta_x = 0$, $\bar{v}_1 = \bar{v}_2 \equiv 0$ and Eq. (2.4.4) reduces to Eq. (2.2.10).

5. Constant temperature wall

$$\bar{T}_1(x,y,t) = T_w, \quad y = -h$$

or Adiabatic wall (2.4.5)

$$\frac{\partial\bar{T}_1(x,y,t)}{\partial y} = 0, \quad y = -h$$

6. Boundary layer edge condition on temperature

$$\overline{T}_2(x,y,t) = T_e, \; y = \delta \tag{2.4.6}$$

7. Energy balance at the interface

$$k_1 \frac{\partial \overline{T}_1}{\partial n} = k_2 \frac{\partial \overline{T}_2}{\partial n} \quad \text{on} \quad y = \eta(x,t)$$

or

$$k_1 \nabla \overline{T}_1 \cdot \overline{n} = k_2 \nabla \overline{T}_2 \cdot \overline{n}$$

where $\overline{n}$ is the unit normal to the interface at any point. Now, if the interface equation is written as

$$F(x,y,t) = y - \eta(x,t) = 0$$

then,

$$\overline{n} = \frac{\nabla F}{|\nabla F|}$$

and hence

$$k_1 \nabla \overline{T}_1 \cdot \nabla F = k_2 \nabla \overline{T}_2 \cdot \nabla F \quad \text{on } F = 0 \quad \text{or} \quad y = \eta(x,t) \tag{2.4.7}$$

Expanding (2.4.7)

$$k_1 \left(\frac{\partial \overline{T}_1}{\partial x} \frac{\partial F}{\partial x} + \frac{\partial \overline{T}_1}{\partial y} \frac{\partial F}{\partial y} \right) = k_2 \left(\frac{\partial \overline{T}_2}{\partial x} \frac{\partial F}{\partial x} + \frac{\partial \overline{T}_2}{\partial y} \frac{\partial F}{\partial y} \right)$$

using the relation $F = y - \eta(x,t)$,

$$k_1\left(-\frac{\partial \bar{T}_1}{\partial x}\eta_x + \frac{\partial \bar{T}_1}{\partial y}\right) = k_2\left(-\frac{\partial \bar{T}_2}{\partial x}\eta_x + \frac{\partial \bar{T}_2}{\partial y}\right)$$

For a flat steady interface $\left(\frac{\partial}{\partial x} \equiv 0,\ \eta_x = 0\right)$ the above equation reduces to Eq. (2.2.13)

8. No temperature jump at the interface

$$\bar{T}_1(x,y,t) = \bar{T}_2(x,y,t) \quad \text{on } y = \eta(x,t) \tag{2.4.8}$$

9. Boundary layer edge condition on pressure

$$\bar{p}_2(x,y,t) = p_e \quad \text{at } y = \delta \tag{2.4.9}$$

10. Balance of normal stresses at the interface

Referring to Fig. 3a it is seen that the discontinuity in the normal stresses across the interface must be balanced by surface tension. Thus

$$\sigma_2 - \sigma_1 = \frac{\Gamma}{R} \quad \text{on } y = \eta(x,t)$$

where Γ is the surface tension and R is the radius of curvature. Now, for a surface that is concave downward, the curvature is given by

$$\frac{1}{R} = \frac{-\eta_{xx}}{(1+\eta_x^2)^{3/2}}$$

Also, as shown in Appendix B, the expression for normal stress is

$$\sigma = -p + \frac{2\mu}{1+\eta_x^2} \frac{\partial u}{\partial x} \eta_x + \frac{\partial v}{\partial y} - \frac{2\mu\eta_x}{1+\eta_x^2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Finally, the normal stress condition becomes

$$\frac{\Gamma\eta_{xx}}{\left(1+\eta_x^2\right)^{1/2}} = (\overline{p}_2 - \overline{p}_1)(1+\eta_x^2) - 2\eta_x^2 \left(\mu_2 \frac{\partial \overline{u}_2}{\partial x} - \mu_1 \frac{\partial \overline{u}_1}{\partial x}\right) - 2\left(\mu_2 \frac{\partial \overline{v}_2}{\partial y} - \mu_1 \frac{\partial \overline{v}_1}{\partial y}\right)$$

$$+ 2\eta_x \, \mu_2 \left(\frac{\partial \overline{u}_2}{\partial y} + \frac{\partial \overline{v}_2}{\partial x}\right) - \mu_1 \frac{\partial \overline{u}_1}{\partial y} + \frac{\partial \overline{v}_1}{\partial x}$$

$$\text{at } y = \eta(x,t) \qquad (2.4.10)$$

In the steady state case Eq. (2.4.10) reduces to Eq. (2.2.16)

The boundary conditions developed so far, (2.4.1) through (2.4.10), are the same as those for the steady-state problem, viz. (2.2.7) through (2.2.16). In the steady-state case there were six unknowns $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{T}_1$, and $\tilde{T}_2$. In the unsteady case, however, there are eight unknowns, $\overline{u}_1$, $\overline{u}_2$, $\overline{p}_1$, $\overline{p}_2$, $\overline{T}_1$, $\overline{T}_2$, $\overline{v}_1$ and $\overline{v}_2$. Thus $\overline{v}_1$ and $\overline{v}_2$ are the two additional unknowns which appear as second derivatives with respect to x and y and also as first derivatives with respect to time (Eq. (2.3.3)). As will be shown in Sec. 2.6 the x and t dependence can be eliminated by assuming a suitable form of interface configuration. Thus one need be concerned only with the boundary conditions on $\overline{v}_1$ and $\overline{v}_2$ in the y direction. Since $\overline{v}_1$ and $\overline{v}_2$ appear as second derivatives w.r.t. y in Eqs. (2.3.3) and (2.3.7), four conditions are required on $\bar{v}_1$ and $\bar{v}_2$. These conditions are, the wall condition on $\bar{v}_1$, the boundary layer edge condition on $\bar{v}_2$ and two interface matching conditions. These four conditions are derived below.

11. No penetration condition at the wall

In the absence of any mass transfer through the wall the no penetration condition is

$$\bar{v}_1(x,y,t) = 0 \ , \quad y = -h \qquad (2.4.11)$$

12. Boundary layer edge condition on the vertical velocity component $\bar{v}_2$

The flow configuration of Fig. 1 behaves as if it is bounded between two walls at $y = -h$ and $y = \delta$ even in the disturbed state, and this would require that the $\bar{v}_2$ be zero at $y = \delta$, i.e.

$$\bar{v}_2(x,y,t) = 0 \ , \ y = \delta \qquad (2.4.12)$$

13. Kinematic condition on $\bar{v}_1$ at the interface

Since there is no mass transfer across the interface, the no penetration condition (commonly referred to as the kinematic condition) at the deformable interface reads

$$\frac{D_1 F}{D_1 t} = 0 \quad \text{on} \quad F = 0 \quad \text{or} \quad y = \eta(x,t)$$

where (2.4.13)

$$\frac{D_1}{D_1 t} \equiv \frac{\partial}{\partial t} + \bar{u}_1 \frac{\partial}{\partial x} + \bar{v}_1 \frac{\partial}{\partial y}$$

14. Kinematic condition on $\overline{v}_2$ at the interface.

Similar reasoning as above gives the no penetration condition on the gas side as

$$\frac{D_2F}{D_2t} = 0 \text{ or } F = 0 \text{ or } y = \eta(x,t)$$

where (2.4.14)

$$\frac{D_2}{D_2t} \equiv \frac{\partial}{\partial t} + \overline{u}_2 \frac{\partial}{\partial x} + \overline{v}_2 \frac{\partial}{\partial y}$$

In the steady-state case the boundary conditions (2.4.11) through (2.4.14) are identically satisfied.

Eqs. (2.3.1) through (2.3.8) supplemented by boundary conditions (2.4.1) through (2.4.14) complete the general formulation of the unsteady problem. It is noted that this problem is highly nonlinear.

2.5 Small Perturbation Formulation of the Unsteady Problem

The solution of the unsteady problem in its most general nonlinear form presents a formidable task and hence a small perturbation solution is attempted. The unsteady motion is viewed as a small perturbation on the steady-state problem of Sec. 2.2. Accordingly, every dependent variable is written as a straight-forward expansion of the form

$$\overline{q}_i(x,y,t) = \tilde{q}_i(y) + s q_{i1}(x,y,t) + s^2 q_{i2}(x,y,t) + --- \qquad (2.5.1)$$

$$i = 1,2$$

where s << 1 is a dimensionless small parameter. It may be recalled here that the steady-state problem has only one independent variable y. Substituting the above expansion into the governing equations (2.3.1) through (2.3.8) it is found that the zeroth order problem is given by the steady-state governing equations (2.2.1) through (2.2.8). The first order problem is given by

Liquid:

1. $$\frac{\partial u_{11}}{\partial x} + \frac{\partial v_{11}}{\partial y} = 0 \tag{2.5.2}$$

2. $$\frac{\partial u_{11}}{\partial t} + \tilde{u}_1 \frac{\partial u_{11}}{\partial x} + \tilde{u}_1' v_{11} = -\frac{1}{\rho_1}\frac{\partial p_{11}}{\partial x} + \nu_1\left(\frac{\partial^2 u_{11}}{\partial x^2} + \frac{\partial^2 u_{11}}{\partial y^2}\right) \tag{2.5.3}$$

3. $$\frac{\partial v_{11}}{\partial t} + \tilde{u}_1 \frac{\partial v_{11}}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_{11}}{\partial y} + \nu_1\left(\frac{\partial^2 v_{11}}{\partial x^2} + \frac{\partial^2 v_{11}}{\partial y^2}\right) \tag{2.5.4}$$

4. $$\frac{\partial T_{11}}{\partial t} + \tilde{u}_1 \frac{\partial T_{11}}{\partial x} + \tilde{T}_1' v_{11} = \frac{k_1}{\rho_1 c_{p1}}\left(\frac{\partial^2 T_{11}}{\partial x^2} + \frac{\partial^2 T_{11}}{\partial y^2}\right) \tag{2.5.5}$$

Gas:

5. $$\frac{\partial u_{21}}{\partial x} + \frac{\partial v_{21}}{\partial y} = 0 \tag{2.5.6}$$

6. $$\frac{\partial u_{21}}{\partial t} + \tilde{u}_2 \frac{\partial u_{21}}{\partial x} + \tilde{u}_2' v_{21} = -\frac{1}{\rho_2}\frac{\partial p_{21}}{\partial x} + \nu_2\left(\frac{\partial^2 u_{21}}{\partial x^2} + \frac{\partial^2 u_{21}}{\partial y^2}\right) \tag{2.5.7}$$

7. $$\frac{\partial v_{21}}{\partial t} + \tilde{u}_2 \frac{\partial v_{21}}{\partial x} = -\frac{1}{\rho}\frac{\partial p_{21}}{\partial y} + \nu_2\left(\frac{\partial^2 v_{21}}{\partial x^2} + \frac{\partial^2 v_{21}}{\partial y^2}\right) \tag{2.5.8}$$

8. $$\frac{\partial T_{21}}{\partial t} + \tilde{u}_2 \frac{\partial T_{21}}{\partial x} + \tilde{T}_2' v_{21} = \frac{k_2}{\rho_2 c_{p2}}\left(\frac{\partial^2 T_{21}}{\partial x^2} + \frac{\partial^2 T_{21}}{\partial y^2}\right) \tag{2.5.9}$$

where primes denote derivatives w.r.t. y.

The next step is to substitute the expansion (2.5.1) into the boundary conditions (2.4.1) through (2.4.14). It may be recalled that the interface boundary conditions (2.4.3), (2.4.4), (2.4.7), (2.4.8), (2.4.10), (2.4.13) and (2.4.14) are applied at the unknown interface $y = \eta(x,t)$. Therefore, consistent with the small perturbation approach, it is assumed that $\eta(x,t)$ is a small disturbance on the steady-state value $\eta = 0$ and that it can be expanded as

$$\eta(x,t) = s\eta_1(x,t) + s^2\eta_2(x,t) + \cdots \qquad (2.5.10)$$

Since $\eta(x,t)$ is small, it is permissible to transfer the above-mentioned boundary conditions from the unknown interface $y = \eta(x,t)$ to the known steady-state value $y = 0$, again consistent with the small perturbation approach. This is accomplished through a Taylor series expansion about $y = 0$,

$$\bar{q}_i(x,y,t) = \bar{q}_i(x,0,t) + \left.\frac{\partial\bar{q}_i}{\partial y}\right|_{y=0} y + \left.\frac{\partial^2\bar{q}_i}{\partial y^2}\right|_{y=0} \frac{y^2}{2!} + \cdots \qquad i = 1,2$$

Or for $y = \eta$

$$\bar{q}_i(x,\eta,t) = \bar{q}_i(x,0,t) + \left.\frac{\partial\bar{q}_i}{\partial y}\right|_{y=0} \eta + \left.\frac{\partial^2\bar{q}_i}{\partial y^2}\right|_{y=0} \frac{\eta^2}{2!} + \cdots \quad i = 1,2 \qquad (2.5.11)$$

where $\eta(x,t)$ is given by (2.5.10).

Substituting Eqs. (2.5.10) and (2.5.11) into the boundary conditions (2.4.1) through (2.4.14) and collecting coefficients of s, it can be verified that the zeroth order problem is given by the steady-state boundary conditions (2.2.7) through (2.2.16). The first order problem is given by

1. $$u_{11}(x,y,t) = 0, \quad y = -h \tag{2.5.12}$$

2. $$u_{21}(x,y,t) = 0, \quad y = \delta \tag{2.5.13}$$

3. $$u_{21} - u_{11} = (\tilde{u}_1' - \tilde{u}_2')\eta_1, \quad y = 0 \tag{2.5.14}$$

4. $$\mu_2\left(\frac{\partial u_{21}}{\partial y} + \frac{\partial v_{21}}{\partial x}\right) - \mu_1\left(\frac{\partial u_{11}}{\partial y} + \frac{\partial v_{11}}{\partial x}\right) = (\mu_1\tilde{u}_1'' - \mu_2\tilde{u}_2'')\eta_1, \quad y = 0 \tag{2.5.15}$$

5. $$T_{11}(x,y,t) = 0, \quad y = -h$$

or

$$\frac{\partial T_{11}}{\partial y}(x,y,t) = 0, \quad y = -h \tag{2.5.16}$$

6. $$T_{21}(x,y,t) = 0, \quad y = \delta \tag{2.5.17}$$

7. $$k_2\frac{\partial T_{21}}{\partial y} - k_1\frac{\partial T_{11}}{\partial y} = (k_1\tilde{T}_1'' - k_2\tilde{T}_2'')\eta_1, \quad y = 0 \tag{2.5.18}$$

8. $$T_{21} - T_{11} = (\tilde{T}_1' - \tilde{T}_2')\eta_1, \quad y = 0 \tag{2.5.19}$$

9. $$p_{21}(x,y,t) = 0, \quad y = \delta \tag{2.5.20}$$

10. $$\Gamma\eta_{1xx} = (p_{21} - p_{11}) + (\tilde{p}_2' - \tilde{p}_1')\eta_1 - 2\left(\mu_2\frac{\partial v_{21}}{\partial y} - \mu_1\frac{\partial v_{11}}{\partial y}\right)\Bigg|_{y=0} \tag{2.5.21}$$

11. $$v_{11}(x,y,t) = 0, \; y = -h \tag{2.5.22}$$

12. $$v_{21}(x,y,t) = 0, \; y = \delta \tag{2.5.23}$$

13. $$\eta_{1t} + \eta_{1x}\tilde{u}_1 - v_{11} = 0, \; y = 0 \tag{2.5.24}$$

14. $$\eta_{1t} + \eta_{1x}\tilde{u}_2 - v_{21} = 0, \; y = \tag{2.5.25}$$

An inspection of Eqs. (2.5.2) through (2.5.9) and (2.5.12) through (2.5.25) makes it clear that the small perturbation assumption results in linearization of the unsteady problem.

2.6 Travelling Wave Solution of Small Perturbation Equations

The small perturbation equations (2.5.2) - (2.5.9) and the boundary conditions (2.5.12) - (2.5.25) exhibit two important properties, (i) linearity and (ii) the coefficients of the unknowns and their derivatives are either constants or functions of y at most. The latter property is a consequence of the steady-state parallel flow assumption. These observations suggest a solution by separation of variables of the type

$$q_{i1}(x,y,t) = q_i(y)\, f(x,t), \; i = 1,2 \tag{2.6.1}$$

A convenient functional form of f is chosen in what follows.

In the boundary conditions of Sec. 2.5 there appear terms in η_1 and its derivatives. Now $\eta_1(x,t)$ is an unknown and its suitable form must be assumed subject to the condition of boundedness w.r.t. x and t (see Sec. 2.4). Suppose that initially the interface is disturbed such that it is sinusoidal in form, thus

$$\eta(x,0) = she^{ikx} \tag{2.6.2}$$

where $s<<1$ is the dimensionless small parameter encountered earlier in Sec. 2.5 and h is the liquid depth. Eq. (2.6.2) implies that $\eta/h << 1$, i.e. the amplitude of the sinusoidal disturbance on the interface is much smaller compared to the liquid depth. k is the wave number of the disturbance given by

$$k = \frac{2\pi}{\lambda} \tag{2.6.3}$$

where λ is the disturbance wavelength.

Now, Eq. (2.6.2) suggests a travelling waveform for $\eta(x,t)$ -

$$\eta(x,t) = she^{i(kx-\omega t)} = she^{ik(x-ct)} \tag{2.6.4}$$

where

$$c = \frac{\omega}{k} \tag{2.6.5}$$

is the speed of propagation of the wave disturbance and ω is the frequency.

It is clear from the comparison of Eqs. (2.6.4) and (2.5.10) that

$$\eta_1(x,t) = he^{i(kx-\omega t)} \tag{2.6.6}$$

It is now obvious that in Eq. (2.6.1)

$$f(x,t) = e^{i(kx-\omega t)}$$

resulting in the solution form

$$q_{i1}(x,y,t) = q_i(y)e^{i(kx-\omega t)} \quad i = 1,2 \tag{2.6.7}$$

It may be mentioned at this point that the work of Secs. 2.5 and 2.6 is equivalent to assuming the following form of solution from the outset

$$\overline{q}_i(x,y,t) = \tilde{q}_i(y) + sq_i(y)e^{i(kx-\omega t)} + 0(s^2) \quad i = 1,2 \tag{2.6.8}$$

Substitution of Eqs. (2.6.6) and (2.6.7) into the governing equations (2.5.2) - (2.5.9) gives the result

Liquid:

1. $$iku_1 + v_1' = 0 \tag{2.6.9}$$

2. $$-i\omega u_1 + iku_1\tilde{u}_1 + v_1\tilde{u}_1' = -\frac{ikp_1}{\rho_1} + \nu_1(u_1'' - k^2u_1) \tag{2.6.10}$$

3. $$-i\omega v_1 + ikv_1\tilde{u}_1 = -\frac{p_1'}{\rho_1} + \nu_1(v_1'' - k^2v_1) \tag{2.6.11}$$

4. $$-i\omega T_1 + ikT_1\tilde{u}_1 + v_1\tilde{T}_1' = \frac{k_1}{\rho_1c_{p1}}(T_1'' - k^2T_1) \tag{2.6.12}$$

Gas:

5. $$iku_2 + v_2' = 0 \tag{2.6.13}$$

6. $$-i\omega u_2 + iku_2\tilde{u}_2 + v_2\tilde{u}_2' = -\frac{ikp_2}{\rho_2} + \nu_2(u_2'' - k^2u_2) \tag{2.6.14}$$

7. $$-i\omega v_2 + ikv_2\tilde{u}_2 = -\frac{p_2'}{\rho_2} + \nu_2(v_2'' - k^2v_2) \tag{2.6.15}$$

8. $$-i\omega T_2 + ikT_2\tilde{u}_2 + v_2\tilde{T}_2' = \frac{k_2}{\rho_2c_{p2}}(T_2'' - k^2T_2) \tag{2.6.16}$$

Similarly, substitution of Eqs. (2.6.6) and (2.6.7) into the boundary conditions (2.5.12) - (2.5.25) gives the result

1. $$u_1(y) = 0, \; y = -h \tag{2.6.17}$$

2. $$u_2(y) = 0, \; y = \delta \tag{2.6.18}$$

3. $$u_2 - u_1 = (\tilde{u}_1' - \tilde{u}_2')h, \; y = 0 \tag{2.6.19}$$

4. $$\mu_2(u_2' + ikv_2) - \mu_1(u_1' + ikv_1) = (\mu_1\tilde{u}_1'' - \mu_2\tilde{u}_2'')h, \; y = 0 \tag{2.6.20}$$

5. $$\begin{gathered} T_1(y) = 0, \; y = -h \\ \text{or} \\ T_1'(y) = 0, \; y = -h \end{gathered} \tag{2.6.21}$$

6. $$T_2(y) = 0, \; y = \delta \tag{2.6.22}$$

7. $$k_2T_2' - k_1T_1' = (k_1\tilde{T}_1'' - k_2\tilde{T}_2'')h, \; y = 0 \tag{2.6.23}$$

8. $$T_2 - T_1 = (\tilde{T}_1' - \tilde{T}_2')h, \; y = 0 \tag{2.6.24}$$

9. $$p_2(y) = 0, \; y = \delta \tag{2.6.25}$$

10. $$\Gamma k^2h + (\tilde{p}_2' - \tilde{p}_1')h + (p_2 - p_1) - 2(\mu_2v_2' - \mu_1v_1'), \quad y = 0 \tag{2.6.26}$$

11. $$v_1(y) = 0, \; y = -h \tag{2.6.27}$$

12. $$v_2(y) = 0, \; y = \delta \tag{2.6.28}$$

13. $$ikh(\tilde{u}_1 - \frac{\omega}{k}) - v_1 = 0 \tag{2.6.29}$$

14. $$ikh(\tilde{u}_2 - \frac{\omega}{k}) - v_2 = 0 \qquad (2.6.30)$$

Thus the original unsteady problem in three independent variables x, y and t has been reduced to a problem in one independent variable y. The total order of the system of Eqs. (2.6.9) - (2.6.16) is 14 and there are 14 boundary conditions (2.6.17) - (2.6.30), thus the resulting problem is mathematically well-posed.

2.7 Reduction to Four Dependent Variables

There are eight unknowns u_1, u_2, p_1, p_2, T_1, T_2, v_1, v_2 in the governing equations and boundary conditions of the previous section. Following the standard procedure in boundary layer stability theory p_1 is eliminated between Eqs. (2.6.10) and (2.6.11) by differentiation and u_1 is eliminated from the resulting equation through Eq. (2.6.9). This manipulation results in a single ordinary differential equation in v_1. Identical operations on the gas side equations (2.6.13) - (2.6.15) yield an ordinary differential equation in v_2. Thus the system of governing equation reduces to the following --

Liquid:

1. $$v_1^{iv} - 2k^2 v_1'' + k^4 v_1 = \frac{ik}{\nu_1}\left[(v_1'' - k^2 v_1)(\tilde{u}_1 - \omega/k) - \tilde{u}_1'' v_1\right] \qquad (2.7.1)$$

2. $$-i\omega T_1 + ikT_1\tilde{u}_1 + v_1\tilde{T}_1' = \frac{k_1}{\rho_1 c_{p1}}(T_1'' - k^2 T_1) \qquad (2.7.2)$$

3. $$v_2^{iv} - 2k^2 v_2 + k^4 v_2 = \frac{ik}{\nu_2}\left[(v_2 - k^2 v_2)(\tilde{u}_2 - \omega/k) - \tilde{u}_2'' v_2\right] \qquad (2.7.3)$$

4. $$-i\omega T_2 + ikT_2\tilde{u}_2 + v_2\tilde{T}_2' = \frac{k_2}{\rho_2 c_{p2}}(T_2'' - k^2 T_2) \qquad (2.7.4)$$

Eqs. (2.7.1) and (2.7.3) are the well-known Orr-Sommerfeld equations. Notice that the order of the system has been reduced from 14 to 12 due to elimination of p_1 and p_2.

In boundary conditions (2.6.17) - (2.6.30) u_1, u_2, p_1 and p_2 are eliminated using Eqs. (2.6.9), (2.6.10), (2.6.13) and (2.6.14). For instance, solving for u_1 and u_2 from Eqs. (2.6.9) and (2.6.13),

$$u_1 = -\frac{v_1'}{ik} \qquad (2.7.5)$$

$$u_2 = -\frac{v_2'}{ik} \qquad (2.7.6)$$

Solving (2.6.10) for p_1 and using (2.7.5)

$$p_1 = \frac{\mu_1}{k^2}(v_1''' - k^2 v_1') + \frac{\rho_1 v_1'}{ik}(\tilde{u}_1 - \omega/k) - \frac{\rho_1 v_1 \tilde{u}_1'}{ik} \qquad (2.7.7)$$

Similarly, from (2.6.13) and (2.6.14)

$$p_2 = \frac{\mu_2}{k^2}(v_2''' - k^2 v_2') + \frac{\rho_2 v_2'}{ik}(\tilde{u}_2 - \omega/k) - \frac{\rho_2 v_2 \tilde{u}_2'}{ik} \qquad (2.7.8)$$

Substituting Eqs. (2.7.5) - (2.7.8) into the boundary conditions (2.6.17) - (2.6.30) the results are

1. $$v_1' = 0, \; y = -h \qquad (2.7.9)$$

2. $$v_2' = 0, \; y = \delta \qquad (2.7.10)$$

3. $$v_1' - v_2' = ikh(\tilde{u}_1' - \tilde{u}_2'),\ y = 0 \tag{2.7.11}$$

4. $$\mu_1(v_1'' + k^2 v_1) - \mu_2(v_2'' + k^2 v_2) = ikh(\mu_1 \tilde{u}_1'' - \mu_2 \tilde{u}_2''),\ y = 0 \tag{2.7.12}$$

5. $$T_1 = 0,\ y = -h$$

or (2.7.13)

$$T_1' = 0,\ y = -h$$

6. $$T_2 = 0,\ y = \delta \tag{2.7.14}$$

7. $$k_2 T_2' - k_1 T_1' = (k_1 \tilde{T}_1'' - k_2 \tilde{T}_2''),\ y = 0 \tag{2.7.15}$$

8. $$T_2 - T_1 = (\tilde{T}_1' - \tilde{T}_2')h,\ y = 0 \tag{2.7.16}$$

9. $$\Gamma k^2 h + h(\tilde{p}_2' - \tilde{p}_1') + \frac{1}{k^2}[\mu_2(v_2''' - k^2 v_2') - \mu_1(v_1''' - k^2 v_1')]$$
$$+ \frac{i}{k}[\rho_2\{v_2 \tilde{u}_2' - v_2'(\tilde{u}_2 - \omega/k)\} - \rho_1\{v_1 \tilde{u}_1' - v_1'(\tilde{u}_1 - \omega/k)\}]$$
$$- 2(\mu_2 v_2' - \mu_1 v_1') = 0 \text{ at } y = 0 \tag{2.7.17}$$

10. $$v_1 = 0,\ y = -h \tag{2.7.18}$$

11. $$v_2 = 0,\ y = \delta \tag{2.7.19}$$

12. $$v_1 - ikh(\tilde{u}_1 - \omega/k) = 0,\ y = 0 \tag{2.7.20}$$

13. $$v_2 - ikh(\tilde{u}_2 - \omega/k) = 0,\ y = 0 \tag{2.7.21}$$

Several important observations can be made at this stage.

(i) The number of boundary conditions has gone down from

14 to 13. This is because p_2 no longer appears as an unknown in Eqs. (2.7.1) - (2.7.4), consequently the boundary condition (2.6.25) is superfluous. Thus for a 12th order system there are 13 boundary conditions.

(ii) The governing equations (2.7.1) - (2.7.4) are homogeneous in v_1, T_1, v_2 and T_2. The boundary conditions (2.7.9) - (2.7.21), however, are not all homogeneous; e.g. Eqs. (2.7.11), (2.7.17), (2.7.20) and (2.7.21). This fact is very significant.

(iii) It is possible to solve for v_1 and v_2 independent of T_1 and T_2. For instance, Eqs. (2.7.1) and (2.7.3) can be solved subject to the nine boundary conditions (2.7.9) - (2.7.12) and (2.7.17) - (2.7.21). This means that the energy equation is decoupled from the equations of motion. It will be shown in Chapter III that such decoupling is not possible when there is mass transfer across the interface. Once a solution for v_1 and v_2 is obtained u_1, u_2 and p_1, p_2 can be obtained from Eqs. (2.7.5) - (2.7.8), if necessary.

(iv) Suppose Eqs. (2.7.1) and (2.7.3) are solved subject to the eight boundary conditions (2.7.9) - (2.7.12) and (2.7.17) - (2.7.20) to obtain a general solution of the form --

$$v_1 = g_1[y;\rho_1,\rho_2,\mu_1,\mu_2,h,\delta,\Gamma,u_e,\omega,k] = g_1[\rho_1,\rho_2,\mu_1,\mu_2,h,\delta,\Gamma,u_e,\omega,k]$$

at the interface

and

$$v_2 = g_2[y;\rho_1,\rho_2,\mu_1,\mu_2,h,\delta,\Gamma,u_e,\omega,k] = g_2[\rho_1,\rho_2,\mu_1,\mu_2,h,\delta,\Gamma,u_e,\omega,k]$$

at the interface

All the arguments of g_1 and g_2 except ω and k are fluid properties and remnants of upstream history (h, δ and u_e) which are known for a given problem. However, ω and k are not independent -- they are connected through the last boundary condition, Eq. (2.7.21). Thus this equation is like a characteristic or frequency equation and in dimensional form it is written as

$$G(\rho_1,\rho_2,\mu_1,\mu_2,h,\delta,\Gamma,u_e,\omega,k) = 0 \qquad (2.7.22)$$

In the present work, G will be called the characteristic function. It is clear from (2.7.22) that given k, ω is uniquely determined (there may be more than one value of ω) for a given set of parameters and vice versa. In this sense the present problem is an eigenvalue problem.

(v) Eq. (2.7.22) suggests the following method for investigating the stability of the interface. Suppose that it is desired to know whether the interface is stable with respect to a disturbance of wavelength λ. Given λ (and hence k) it is possible, in principle, to determine a set of values of ω. These values of ω will be, in general, complex. Let $\omega = \omega_r + i\omega_i$. Then the equation of the interface (2.6.4) becomes

$$\eta(x,t) = she^{\omega_i t}e^{i(kx-\omega_r t)} \qquad (2.7.23)$$

Hence

If $\omega_i > 0$ interface amplitude will grow exponentially with time, i.e., unstable interface

If $\omega_i < 0$ interface amplitude will decay exponentially with time, i.e., stable interface

If $\omega_i = 0$ interface amplitude remains constant , i.e., stable interface

Let Eq. (2.7.23) be rewritten as

$$\eta(x,t) = she^{kc_i t}e^{ik(x-c_r t)} \tag{2.7.24}$$

where

$$c = \omega/k \tag{2.7.25}$$

Thus c_r has the dimension of speed and it is referred to as the phase speed. c_i appears in the amplitude term and is called the amplification factor.

2.8 Non-dimensionalization of the Eigenvalue Problem

The vertical co-ordinates in the liquid and gas are non-dimensionalized with respect to the liquid depth and the boundary layer thickness respectively. Thus

$$\xi = \frac{y}{h} \tag{2.8.1}$$

$$\eta = \frac{y}{\delta} \tag{2.8.2}$$

The velocities on the liquid side are made dimensionless relative to the interface velocity u_{if} in Eq. (2.2.25) and the gas side velocities are made dimensionless with respect to the edge velocity u_e. Hence the steady-state velocity profiles now have the form

Liquid velocity profile:

$$\hat{u}_1(\xi) = 1 + \xi \qquad -1 \leq \xi \leq 0 \tag{2.8.3}$$

Gas velocity profile:

$$\hat{u}_2(\eta) = \frac{\eta + \varepsilon\bar{\mu}}{1+\varepsilon\bar{\mu}} \qquad 0 \leq \eta \leq 1 \tag{2.8.4}$$

The interface velocity, non-dimensionalized with respect to boundary layer edge velocity, is given by

$$\bar{u} = \frac{\varepsilon\bar{\mu}}{1+\varepsilon\bar{\mu}} \tag{2.8.5}$$

where the non-dimensional thickness and viscosity ratios ε and $\bar{\mu}$ are given by

$$\varepsilon = \frac{h}{\delta} \tag{2.8.6}$$

$$\bar{\mu} = \frac{\mu_2}{\mu_1} \tag{2.8.7}$$

The next step is to non-dimensionalize the governing equations (2.7.1) and (2.7.3) and the boundary conditions (2.7.9) - (2.7.12) and (2.7.17) - (2.7.21). To this end the following quantities are introduced

$$\psi_1 = \frac{v_1}{u_{if}} \tag{2.8.8}$$

$$\psi_2 = \frac{v_2}{u_e} \tag{2.8.9}$$

Then using Eq. (2.8.1) and (2.8.2) the Orr-Sommerfeld equations (2.7.1) and (2.7.3) become (for a linear velocity profile)

$$\psi_1^{iv} - 2\alpha_1^2\psi_1'' + \alpha_1^4\psi_1 = i\alpha_1 R_1(\psi_1'' - \alpha_1^2\psi_1)(\hat{u}_1 - c_1) \tag{2.8.10}$$

and

$$\ddot{\ddot{\psi}}_2 - 2\alpha_2^2\ddot{\psi}_2 + \alpha_2^4\psi_2 = i\alpha_2 R_2(\ddot{\psi}_2 - \alpha_2^2\psi_2)(\hat{u}_2 - c_2) \tag{2.8.11}$$

where primes and dots denote differentiation w.r.t. ξ and η respectively.

dimensionless disturbance wave number in the liquid

$$\alpha_1 = kh \tag{2.8.12}$$

liquid Reynolds number

$$R_1 = \frac{u_{if}h}{\nu_1} \tag{2.8.13}$$

dimensionless phase speed in the liquid

$$c_1 = \frac{\omega/k}{u_{if}} \tag{2.8.14}$$

dimensionless disturbance wave number in the gas

$$\alpha_2 = k\delta \tag{2.8.15}$$

gas Reynolds number

$$R_2 = \frac{u_e \delta}{\nu_2} \tag{2.8.16}$$

dimensionless phase speed in the gas

$$c_2 = \frac{\omega/k}{u_e} \tag{2.8.17}$$

The following relationships exist between $\alpha_1, \alpha_2, c_1, c_2$, and R_1, R_2:

$$\alpha_1 = \varepsilon\alpha_2 \tag{2.8.18}$$

$$c_2 = \bar{u}c_1 \tag{2.8.19}$$

$$R_1 = \frac{\bar{u}\bar{\varepsilon}\bar{\mu}}{\bar{\rho}} R_2 \tag{2.8.20}$$

The boundary conditions (2.7.9) - (2.7.12) and (2.7.17) - (2.7.21) assume the following form for a linear velocity profile.

1. $$\psi_1'(\xi) = 0, \quad \xi = -1 \tag{2.8.21}$$

2. $$\dot{\psi}_2(\eta) = 0, \quad \eta = 1 \tag{2.8.22}$$

3. $$\bar{u}[\psi_1'(\xi) - i\alpha_1\hat{u}_1'] = \varepsilon[\dot{\psi}_2(\eta) - i\alpha_1\dot{\hat{u}}_2] \text{ at } \xi = 0,\ \eta = 0 \tag{2.8.23}$$

4. $$\bar{u}[\psi_1''(\xi) + \alpha_1^2\psi_1(\xi)] = \bar{\mu}\varepsilon^2[\ddot{\psi}_2(\eta) + \alpha_2^2\psi_2(\eta)] \text{ at } \xi = \eta = 0 \tag{2.8.24}$$

5. $$\frac{1}{\alpha_2^2 R_2}\left\{\dddot{\psi}_2(\eta) - \alpha^2\dot{\psi}_2(\eta)\right\} - \frac{\bar{u}^2}{\bar{\rho}}\frac{1}{\alpha_1^2 R_1}\left\{\psi_1'''(\xi) - \alpha_1^2\psi_1'(\xi)\right\}$$

$$+ \frac{i}{\alpha_2}\left\{\dot{\hat{u}}_2\psi_2(\eta) - (\hat{u}_2(\eta) - c_2)\dot{\psi}_2(\eta)\right\} - \frac{\bar{u}^2}{\bar{\rho}}\frac{i}{\alpha_1}\left\{\hat{u}_1'\psi_1(\xi) - (\hat{u}_1(\xi) - c_1)\psi_1'(\xi)\right\}$$

$$-\frac{2}{\varepsilon\bar{\mu}R_2}\left\{\varepsilon\bar{\mu}\dot{\psi}_2(\eta) - \bar{u}\psi_1'(\xi)\right\} = -\frac{\bar{u}^2}{\bar{\rho}}(\alpha_1^2 W^2 + \frac{1}{F^2}) \qquad (2.8.25)$$

6. $$\psi_1(\xi) = 0, \ \xi = -1 \qquad (2.8.26)$$

7. $$\psi_2(\eta) = 0, \ \eta = 1 \qquad (2.8.27)$$

8. $$\psi_1(\xi) - i\alpha_1(\hat{u}_1(\xi) - c_1) = 0 \quad \text{at } \xi = 0 \qquad (2.8.28)$$

9. $$\psi_2(\eta) - i\alpha_1(\hat{u}_2(\eta) - c_2) = 0 \quad \text{at } \eta = 0 \qquad (2.8.29)$$

where,

Weber number $W = \sqrt{\dfrac{\Gamma}{\rho u_{if}^{\ 2} h}}$ (2.8.30)

Froude number $F = u_{if}/\sqrt{gh}$ (2.8.31)

CHAPTER III

FORMULATION OF THE MASS TRANSFER PROBLEM

3.1 Simplifying assumptions

The following simplifying assumptions were made for the mass transfer problem in order to obtain a mathematically tractable model.

(i) As shown in Fig. 1b the liquid is injected at $y = -h$. At the interface the liquid vaporizes and the vapor is entrained into the gas boundary layer through convection and diffusion. It is assumed that under steady-state conditions the liquid injection rate exactly balances the loss of liquid species at the interface. This assures that a liquid layer of constant depth 'h' is maintained.

(ii) The liquid layer thickness h and boundary layer thickness δ are assumed to be prescribed by a suitable upstream solution.

(iii) The liquid and gas motion are assumed laminar (or quasi-laminar) and two dimensional both for the steady-state and the unsteady problem. In the latter case only two dimensional disturbances are considered.

(iv) The gas vapor mixture has constant properties such as density, viscosity, thermal conductivity and specific heat. In the case of small rates of mass transfer, which is the concern of this work, these properties have nearly the same values as the gas alone.

(v) The Prandtl and Lewis numbers for the gas mixture are unity in both the steady-state and unsteady cases. Thus the velocity, temperature and concentration boundary layers have the same thickness δ.

3.2 The Steady-State Problem

The steady-state or the mean flow is assumed to be incompressible and parallel with uniform injection rate at the wall and uniform evaporative mass transfer at the interface (Fig. 1b). The mass transfer rate is assumed to be consistent with the thermodynamic conditions and its determination is described in Sec. 3.3. The governing equations are

Liquid:

1. Continuity

$$\frac{d\tilde{v}_1}{dy} = 0 \tag{3.2.1}$$

2. x-momentum

$$\tilde{v}_1 \frac{d\tilde{u}_1}{dy} = \nu_1 \frac{d^2\tilde{u}_1}{dy^2} \tag{3.2.2}$$

3. y-momentum

$$\frac{d\tilde{p}_1}{dy} = - \rho_1 g \tag{3.2.3}$$

4. Energy

$$\tilde{v}_1 \frac{d\tilde{T}_1}{dy} = \frac{k_1}{\rho c_{p1}} \frac{d^2\tilde{T}_1}{dy^2} \tag{3.2.4}$$

Gas-vapor boundary layer:

5. Continuity

$$\frac{d\tilde{v}_2}{dy} = 0 \tag{3.2.5}$$

6. x-momentum

$$\tilde{v}_2 \frac{d\tilde{u}_2}{dy} = \nu_2 \frac{d^2\tilde{u}_2}{dy^2} \tag{3.2.6}$$

7. y-momentum

$$\frac{d\tilde{p}_2}{dy} \simeq 0 \tag{3.2.7}$$

8. Energy (neglecting viscous dissipation and pressure gradient terms)

$$\tilde{v}_2 \frac{d\tilde{T}_2}{dy} = \frac{k_2}{\rho_2 c_{p2}} \frac{d^2\tilde{T}_2}{dy^2} \tag{3.2.8}$$

9. Species continuity

$$\tilde{v}_2 \frac{d\tilde{\chi}}{dy} = D \frac{d^2\tilde{\chi}}{dy^2} \tag{3.2.9}$$

where $\tilde{\chi}$ is the mass fraction of vapor in the gas boundary layer. The total order of the system of Equations (3.2.1) - (3.2.9) is 14 and an equal number of boundary conditions must be provided. These conditions are listed below and their explanation thereafter.

1. No slip at the wall

$$\tilde{u}_1(-h) = 0 \tag{3.2.10}$$

2. Boundary layer edge condition on the velocity

$$\tilde{u}_2(\delta) = u_e \tag{3.2.11}$$

3. No slip in the tangential velocity at the interface

$$\tilde{u}_1(0) = \tilde{u}_2(0) \tag{3.2.12}$$

4. Balance of shear stresses at the interface

$$\mu_1 \left.\frac{d\tilde{u}_1}{dy}\right|_{y=0} = \mu_2 \left.\frac{d\tilde{u}_2}{dy}\right|_{y=0} \tag{3.2.13}$$

5. Constant temperature or adiabatic wall

$$\tilde{T}_1(-h) = Tw \text{ or } \left.\frac{d\tilde{T}_1}{dy}\right|_{y=-h} = 0 \tag{3.2.14}$$

6. Boundary layer edge condition on temperature

$$\tilde{T}_2(\delta) = T_e \tag{3.2.15}$$

7. Energy balance at the interface

Heat transferred from the gas to the interface is partly conducted through the liquid and the remainder is spent in vaporizing the liquid.

$$k_1 \frac{d\tilde{T}_1}{dy} = k_2 \frac{d\tilde{T}_2}{dy} + \rho_1 \tilde{v}_1 \ell \quad \text{at } y = 0 \tag{3.2.16}$$

where ℓ is the latent heat of vaporization of liquid.

8. No jump in temperature at the interface

$$\tilde{T}_1(0) = \tilde{T}_2(0) \tag{3.2.17}$$

9. Boundary layer edge condition on pressure

$$\tilde{p}_2(\delta) = p_e \tag{3.2.18}$$

10. Balance of normal stresses at the interface

$$\tilde{p}_1 + \rho_1\tilde{v}_1^2 = \tilde{p}_2 + \rho_2\tilde{v}_2^2 \quad \text{at } y = 0 \tag{3.2.19}$$

11. Specified injection velocity at the wall

$$\tilde{v}_1(-h) = \dot{m}/\rho_1 \tag{3.2.20}$$

where $\dot{m}$ is liquid mass transfer rate (mass injected per unit time per unit area) at the wall.

12. Global mass balance at the interface

$$\rho_1\tilde{v}_1 = \rho_2\tilde{v}_2 \quad \text{at } y = 0 \tag{3.2.21}$$

13. Balance of liquid species across the interface

$$\rho_1\tilde{v}_1 = \rho_2\tilde{v}_2\tilde{\chi} - \rho_2 D \frac{d\tilde{\chi}}{dy} \tag{3.2.22}$$

14. Boundary layer edge condition on vapor concentration

$$\tilde{\chi}(\delta) = 0 \tag{3.2.23}$$

Note that in Eqs. (3.2.9) and (3.2.22), $D = \nu$ since $Pr_2 = Le_2 = 1$

The first ten boundary conditions (3.2.10) - (3.2.19) are modified forms of Eqs. (2.2.7) - (2.2.16) to account for mass transfer. The value of mass flux in at the wall is determined by the thermodynamic conditions of the problem (Sec. 3.3). Eq. (3.2.21) states that the mass flux of liquid reaching the interface is balanced by the gas-vapor mass flux leaving the interface (note that subscript '2' now denotes the gas-vapor mixture). Eq. (3.2.2) expresses the fact that the mass flux of liquid species at the interface is balanced by the convection ($\rho_2 \tilde{v}_2 \tilde{\chi}$) and diffusion ($-\rho_2 D \, \partial\tilde{\chi}/\partial y$) of the vapor species. If a similar condition is written for the gas species at the interface, assuming that the air is insoluble in the liquid (i.e. zero mass flux of air at the interface),

$$0 = \rho_2 \tilde{v}_2 \tilde{\chi}_g - \rho_2 D \frac{d\tilde{\chi}_g}{dy} \tag{3.2.24}$$

where $\tilde{\chi}_g$ is the mass fraction of gas species. Since $\tilde{\chi} = 1 - \tilde{\chi}_g$, the above equation can be written in terms of $\tilde{\chi}$ as,

$$\rho_2 \tilde{v}_2 = \rho_2 \tilde{v}_2 \chi - \rho_2 D \frac{d\tilde{\chi}}{dy} \tag{3.2.25}$$

when Eq. (3.2.25) is combined with (3.2.22) the result is Eq. (3.2.21). Thus the latter equation can be said to express the condition of insolubility of the gas species in the liquid. Finally, the edge condition or concentration (3.2.23) could well be $\tilde{\chi}(\delta) = \chi_0$, where χ_0 is some prescribed vapor mass fraction in the inviscid free stream. Thus $\tilde{\chi}$ in the present formulation could be treated as a 'reduced' concentration.

The solution of the steady-state problem with mass transfer (as a function of $\dot{m}$) is —

v-component profile in liquid

$$\tilde{v}_1 = \dot{m}/\rho_1 = \text{constant} \tag{3.2.26}$$

v-component profile in gas-vapor

$$\tilde{v}_2 = \dot{m}/\rho_2 = \text{constant} \tag{3.2.27}$$

u-component profile in liquid

$$\frac{\tilde{u}_1}{u_e} = \frac{\exp(\dot{m}y/\mu_1) - \exp(-\dot{m}h/\mu_1)}{\exp(\dot{m}\delta/\mu_2) - \exp(-\dot{m}h/\mu_1)} \tag{3.2.28}$$

u-component profile in gas-vapor

$$\frac{\tilde{u}_2}{u_e} = \frac{\exp(\dot{m}y/\mu_2) - \exp(-\dot{m}h/\mu_1)}{\exp(\dot{m}\delta/\mu_2) - \exp(-\dot{m}h/\mu_1)} \tag{3.2.29}$$

pressure profile in liquid

$$\tilde{p}_1 = p_e + \dot{m}^2\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) - \rho_1 gy \tag{3.2.30}$$

pressure profile in gas-vapor

$$\tilde{p}_2 = p_e = \text{constant} \tag{3.2.31}$$

temperature profile in liquid

$$\frac{\tilde{T}_1}{T_e} = \frac{\frac{c_{p2}}{c_{p1}}\left[\exp(\dot{m}yPr_1/\mu_1) - \exp(-\dot{m}hPr_1/\mu_1)\right] - \frac{c_{p2}}{c_{p1}}\frac{T_w}{T_e}\left[\exp(\dot{m}yPr_1/\mu_1) - 1\right] + \{\exp(\dot{m}\delta Pr_2/\mu_2) - 1\}\left[\frac{T_w}{T_e} - \frac{\ell}{c_{p1}T_e}\{\exp(\dot{m}yPr_1/\mu_1) - \exp(\dot{m}hPr_1/\mu_1)\}\right]}{\{\exp(\dot{m}\delta Pr_2/\mu_2) - 1\} + \frac{c_{p2}}{c_{p1}}\{1 - \exp(-\dot{m}hPr_1/\mu_1)\}}$$

constant temperature wall

$$= 1 - \frac{\ell}{c_{p2}T_e}\{\exp(\dot{m}\delta/\mu_2) - 1\} = \text{const.} \qquad (3.2.32)$$

adiabatic wall

temperature profile in gas

$$\frac{\tilde{T}_2}{T_e} = \frac{\{\exp(\dot{m}yPr_2/\mu_2) - 1\} + \frac{c_{p2}}{c_{p1}}\{1 - \exp(-\dot{m}hPr_1/\mu_1)\} + \{\exp(\dot{m}\delta Pr_2/\mu_2) - \exp(\dot{m}yPr_2/\mu_2)\}\left[\frac{T_w}{T_e} - \frac{\ell}{c_{p1}T_e} 1 - \exp(\dot{m}hPr_1/\mu_1)\right]}{\{\exp(\dot{m}\delta Pr_2/\mu_2) - 1\} + \frac{c_{p2}}{c_{p1}}\{1 - \exp(\dot{m}hPr_1/\mu_1)\}}$$

constant temperature wall

$$= 1 - \frac{\ell}{c_{p2}T_e}\{\exp(\dot{m}\delta/\mu_2) - \exp(\dot{m}y/\mu_2)\} \qquad (3.2.33)$$

adiabatic wall

where

$$Pr_1 = \mu_1 c_{p1}/k_1 = \text{Liquid Prandtl Number}$$

and

$$Pr_2 = \mu_2 c_{p2}/k_2 = \text{Gas Prandtl Number} = 1$$

Vapor mass fraction profile

$$\tilde{\chi} = 1 - \exp(\dot{m}y/\mu_2)/\exp(\dot{m}\delta/\mu_2) \tag{3.2.34}$$

The interface quantities are obtained from Eqs. (3.2.26) - (3.2.34) by putting $y = 0$. Thus

Interface velocity - v component

$$\begin{aligned} v_{if} &= \dot{m}/\rho_1 \quad y = 0^- \\ &= \dot{m}/\rho_2 \quad y = 0^+ \end{aligned} \tag{3.2.35}$$

Interface velocity - u component

$$\frac{u_{if}}{u_e} = \frac{1 - \exp(-\dot{m}h/\mu_1)}{\exp(\dot{m}\delta/\mu_2) - \exp(\dot{m}h/\mu_1)} \tag{3.2.36}$$

Interface temperature

$$\frac{T_{if}}{T_e} = \frac{\dfrac{c_{p2}}{c_{p1}}\{1 - \exp(-\dot{m}hPr_1/\mu_1)\} + \{\exp(\dot{m}\delta Pr_2/\mu_2) - 1\}\left[\dfrac{T_w}{T_e} - \dfrac{\ell}{c_{p1}T_e}\{1 - \exp(-\dot{m}hPr_1/\mu_1)\}\right]}{\{\exp(\dot{m}\delta Pr_2/\mu_2) - 1\} + \dfrac{c_{p2}}{c_{p1}}\{1 - \exp(-\dot{m}hPr_1/\mu_1)\}} \tag{3.2.37}$$

Finally, one important fact needs to be brought to the attention of the reader. In the case of the gas boundary layer the conditions $\tilde{u}_2 = u_e$, $\tilde{T}_2 = T_e$ and $\tilde{\chi} = 0$ were applied at $y = \delta$ rather than at $y \to \infty$. This was done in order to obtain bounded solutions. Consequently, these solutions have discontinuous first derivatives at the edge of the boundary layer.

3.3 Determination of Mass Transfer Rate $\dot{m}$

It was mentioned in the previous section that the mass flux $\dot{m}$ is determined by thermodynamic conditions. This task is accomplished as follows.

It has been assumed so far that $\dot{m}$ is specified and that the liquid depth h remains constant. The latter implies that whatever amount of injected liquid reaches the interface must vaporize and then convect and diffuse into the gas. Now Eq. (3.2.34) shows that at the interface the vapor has a definite concentration under steady-state conditions. If the vapor alone were to occupy a unit volume above the interface it will be in phase equilibrium with the liquid at the interface temperature and pressure. Hence this 'saturation' condition fixes the partial pressure of the vapor at the interface temperature. The partial pressure of the vapor, in turn, determines the interface concentration. The phase equilibrium is expressed by the Clausius-Clayperon equation as

$$\tilde{p}_v = Ke^{-\ell/R\tilde{T}_2} \qquad (3.3.1)$$

for a vapor point that behaves like an ideal gas. $\tilde{p}_v$ is the partial pressure of the vapor at temperature $\tilde{T}_2$. Thus

$$\tilde{p}_v = \tilde{\chi}\tilde{p}_2 \tag{3.3.2}$$

by Dalton's law for an ideal gas. In (3.3.2) $\tilde{\chi}$ is the concentration and $\tilde{p}$ is the mixture pressure. In (3.3.1) K is a constant and ℓ is latent heat of vaporization which is assumed constant. ℓ is independently known from calorimetric data. Combining the last two equations

$$\tilde{\chi}\tilde{p}_2 = Ke^{-\ell/R\tilde{T}_2} \quad \text{at } y = 0 \tag{3.3.3}$$

where $\tilde{T}_2$ is the gas-vapor mixture temperature at the interface.

Using a more convenient form of (3.3.1) the result is

$$\ell n\left(\frac{\tilde{\chi}\tilde{p}_2}{p_{ref}}\right) = -\frac{\ell}{R}\left(\frac{1}{\tilde{T}_2} - \frac{1}{T_{ref}}\right) \quad \text{at } y = 0 \tag{3.3.4}$$

where p_{ref} and T_{ref} are some reference 'saturation' conditions, i.e., T_{ref} is the boiling point at pressure p_{ref}, R is the gas constant of the vapor.

Recalling that the steady-state solutions were obtained in terms of $\dot{m}$, Eq. (3.3.4) becomes

$$\ell n\left(\frac{\tilde{\chi}(\dot{m})\tilde{p}_2(\dot{m})}{p_{ref}}\right) = -\frac{\ell}{R}\left(\frac{1}{\tilde{T}_2(\dot{m})} - \frac{1}{T_{ref}}\right) \quad \text{at } y = 0 \tag{3.3.5}$$

$\dot{m}$ can be determined, in principle, by solution of (3.3.5). For instance, in the case of constant temperature wall, substituting for

$\tilde{p}_2$, $\tilde{T}_2$ and $\tilde{\chi}$ from Eqs. (3.2.31), (3.2.33) and (3.2.34) into Eq. (3.3.5)

$$\frac{\ell}{RT_{ref}} - \ell n\frac{p_e}{p_{ref}} = \ell n\{1 - \exp(-\dot{m}\delta/\mu_2)\} + \frac{\ell}{RT_e} \cdot \frac{N}{D}$$

where

$$N = \exp(\dot{m}\delta Pr_2/\mu_2 - 1) + \frac{c_{p2}}{c_{p1}}\{1 - \exp(-\dot{m}hPr_1/\mu_1)\} \qquad (3.3.6)$$

$$D = \frac{c_{p2}}{c_{p1}}\{1 - \exp(-\dot{m}hp_{r1}/\mu_1)\}$$

$$+ \{\exp(\dot{m}\delta Pr_2/\mu_2) - 1\}\left[\frac{T_w}{T_e} - \frac{\ell}{c_{p1}T_e}\{1 - \exp(-\dot{m}hPr_1/\mu_1)\}\right]$$

with $Pr_2 = 1$

Eq. (3.3.6) is nonlinear (even when simplifications are made for small mass transfer rates) and is solved by the Newton-Raphson method. This procedure is described in Sec. 5.3.

3.4 The Unsteady Problem

The formulation of unsteady problem follows closely the zero mass transfer case. When the steady-state configuration of Fig. 1.1b is disturbed the resulting unsteady, two dimensional, incompressible motion with interface mass transfer is governed by the equations below.

Liquid:

1. Continuity

$$\frac{\partial \tilde{u}_1}{\partial x} + \frac{\partial \tilde{v}_1}{\partial y} = 0 \qquad (3.4.1)$$

2. x-momentum

$$\frac{\partial \bar{u}_1}{\partial t} + \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x} + \bar{v}_1 \frac{\partial \bar{u}_1}{\partial y} = -\frac{1}{\rho_1}\frac{\partial \bar{p}_1}{\partial x} + \nu_1 \left(\frac{\partial^2 \bar{u}_1}{\partial x^2} + \frac{\partial^2 \bar{u}_1}{\partial y^2}\right) \quad (3.4.2)$$

3. y-momentum

$$\frac{\partial \bar{v}_1}{\partial t} + \bar{u}_1 \frac{\partial \bar{v}_1}{\partial x} + \bar{v}_1 \frac{\partial \bar{v}_1}{\partial y} = -\frac{1}{\rho_1}\frac{\partial \bar{p}_1}{\partial y} + \nu_1 \left(\frac{\partial^2 \bar{v}_1}{\partial x^2} + \frac{\partial^2 \bar{v}_1}{\partial y^2}\right) \quad (3.4.3)$$

4. Energy

$$\frac{\partial \bar{T}_1}{\partial t} + \bar{u}_1 \frac{\partial \bar{T}_1}{\partial x} + \bar{v}_1 \frac{\partial \bar{T}_1}{\partial y} = \frac{k_1}{\rho_1 c_{p1}} \left(\frac{\partial^2 \bar{T}_1}{\partial x^2} + \frac{\partial^2 \bar{T}_1}{\partial y^2}\right) \quad (3.4.4)$$

Gas-vapor:

5. Continuity

$$\frac{\partial \bar{u}_2}{\partial x} + \frac{\partial \bar{v}_2}{\partial y} = 0 \quad (3.4.5)$$

6. x-momentum

$$\frac{\partial \bar{u}_2}{\partial t} + \bar{u}_2 \frac{\partial \bar{u}_2}{\partial x} + \bar{v}_2 \frac{\partial \bar{u}_2}{\partial y} = -\frac{1}{\rho_2}\frac{\partial \bar{p}_2}{\partial x} + \nu_2 \left(\frac{\partial^2 \bar{u}_2}{\partial x^2} + \frac{\partial^2 \bar{u}_2}{\partial y^2}\right) \quad (3.4.6)$$

7. y-momentum

$$\frac{\partial \bar{v}_2}{\partial t} + \bar{u}_2 \frac{\partial \bar{v}_2}{\partial x} + \bar{v}_2 \frac{\partial \bar{v}_2}{\partial y} = -\frac{1}{\rho_2}\frac{\partial \bar{p}_2}{\partial y} + \nu_2 \left(\frac{\partial^2 \bar{v}_2}{\partial x^2} + \frac{\partial^2 \bar{v}_2}{\partial y^2}\right) \quad (3.4.7)$$

8. Energy (neglecting viscous dissipation and pressure gradient terms)

$$\frac{\partial \overline{T}_2}{\partial t} + \overline{u}_2 \frac{\partial \overline{T}_2}{\partial x} + \overline{v}_2 \frac{\partial \overline{T}_2}{\partial y} = \frac{k_2}{\rho_2 c_{p2}} \left(\frac{\partial^2 \overline{T}_2}{\partial x^2} + \frac{\partial^2 \overline{T}_2}{\partial y^2} \right) \qquad (3.4.8)$$

9. Continuity of vapor species

$$\frac{\partial \overline{\chi}}{\partial t} + \overline{u} \frac{\partial \overline{\chi}}{\partial x} + \overline{v} \frac{\partial \overline{\chi}}{\partial y} = D \left(\frac{\partial^2 \overline{\chi}}{\partial x^2} + \frac{\partial^2 \overline{\chi}}{\partial y^2} \right) \qquad (3.4.9)$$

This system of equations is to be solved subject to the following boundary conditions. This development follows very closely the work in Sec. 2.4.

1. No slip at the wall

$$\overline{u}_1(x,y,t) = 0, \quad y = -h \qquad (3.4.10)$$

2. Edge condition on the u-velocity component

$$\overline{u}_2(x,y,t) = u_e, \quad y = \delta \qquad (3.4.11)$$

3. No slip in tangential velocity at the interface (Fig. 2) (same as in zero mass transfer case)

$$\overline{u}_2 - \overline{u}_1 = (\overline{v}_1 - \overline{v}_2) \frac{\partial \eta}{\partial x} \text{ at } y = \eta(x,t) \qquad (3.4.12)$$

4. Balance of shear stresses at the interface (no modification over zero mass transfer case)

$$\frac{1-\eta_x^2}{1+\eta_x^2}\mu_2\left(\frac{\partial\bar{u}_2}{\partial y}+\frac{\partial\bar{v}_2}{\partial x}\right)-\frac{2\mu_2\eta_x}{1+\eta_x^2}\left(\frac{\partial\bar{u}_2}{\partial x}-\frac{\partial\bar{v}_2}{\partial y}\right)$$

$$=\frac{1-\eta_x^2}{1+\eta_x^2}\mu_1\left(\frac{\partial\bar{u}_1}{\partial y}+\frac{\partial\bar{v}_1}{\partial x}\right)-\frac{2\mu_1\eta_x}{1+\eta_x^2}\left(\frac{\partial\bar{u}_1}{\partial x}-\frac{\partial\bar{v}_1}{\partial y}\right) \tag{3.4.13}$$

5. Constant temperature wall

$$\bar{T}_1(x,y,t) = T_w,\ y = -h$$

or adiabatic wall (3.4.14)

$$\frac{\partial\bar{T}_1(x,y,t)}{\partial y} = 0,\ y = -h$$

6. Edge condition on temperature

$$\bar{T}_2(x,y,t) = T_e,\ y = \delta \tag{3.4.15}$$

7. Energy balance at the interface

$$k_2\frac{\partial\bar{T}_2}{\partial n} = k_1\frac{\partial\bar{T}_1}{\partial n} + \rho_2(\bar{V}_{R2}\cdot\bar{n})\ell \quad \text{on } y = \eta(x,t)$$

where $\bar{V}_{R2}$ is gas velocity vector relative to the interface and $\bar{n}$ is the outward unit normal to the interface. Since $\bar{V}_{R2} = \bar{V}_2 - \bar{V}_{if}$, the previous equation can be written as

$$k_2\nabla\bar{T}_2\cdot\bar{n} = k_1\nabla\bar{T}_1\cdot\bar{n} + \ell\rho_2(\bar{V}_2 - \bar{V}_{if})\cdot n$$

If $F(x,y,t) = 0$ represents the interface the unit normal is given by

$$\bar{n} = \frac{\nabla F}{|\nabla F|}$$

thus

$$k_2 \nabla \overline{T}_2 \cdot \frac{\nabla F}{|\nabla F|} = \nabla \overline{T}_1 \cdot \frac{\nabla F}{|\nabla F|} + \frac{\ell \rho_2}{|\nabla F|} \left[\overline{V}_2 \cdot \nabla F - \overline{V}_{if} \cdot \nabla F \right]$$

A very delicate argument needs to be made with regard to the term $V_{if} \cdot \nabla F$. At every instant of time, the interface shape is given by a surface formed by those points that have the value $F = 0$ at that instant. Thus, to an observer moving with the interface, there is no change in value of the function F. In other words, the total time rate of change of $F(x,y,t)$ following a point on the surface is zero. Hence,

$$\frac{\partial F}{\partial t} + \overline{V}_{if} \cdot \nabla F = 0$$

or

$$\overline{V}_{if} \cdot \nabla F = - \frac{\partial F}{\partial t}$$

The reader is referred to Karamcheti[43] for a more complete discussion on this point. Eliminating the term $\overline{V}_{if} \cdot \nabla F$ the energy balance condition reads

$$k_2 \nabla \overline{T}_2 \cdot \nabla F = \nabla \overline{T}_1 \cdot \nabla F + \ell \rho_2 (\overline{V}_2 \cdot \nabla F + \frac{\partial F}{\partial t}) = 0$$

However,

$$\frac{\partial F}{\partial t} + \overline{V}_2 \cdot \nabla F = \frac{\partial F}{\partial t} + \overline{u}_2 \frac{\partial F}{\partial x} + \overline{v}_2 \frac{\partial F}{\partial y} = \frac{D_2 F}{D_2 t}$$

and hence,

$$k_2 \nabla \overline{T}_2 \cdot \nabla F = k_1 \nabla \overline{T}_1 \cdot \nabla F + \ell \rho_2 \frac{D_2 F}{D_2 t} \quad \text{on } y = \eta(x,t) \qquad (3.4.16)$$

This equation reduces to Eq. (3.2.16) for steady-state conditions and to Eq. (2.4.7) in the absence of mass transfer.

8. No temperature jump at the interface

$$\overline{T}_1(x,y,t) = \overline{T}_2(x,y,t) \quad \text{on } y = \eta(x,t) \qquad (3.4.17)$$

9. Edge condition on pressure

$$\overline{p}_2(x,y,t) = p_e, \; y = \delta \qquad (3.4.18)$$

10. Balance of normal stresses (momentum flux) at the interface (Fig. 3b)

This condition is derived by applying Newton's second law to the fluid crossing the interface, viz.

Rate of change of normal momentum per unit area

= External stress in normal direction

Normal momentum flux above the interface $= \left[(\rho_2 \overline{V}_{R2} \cdot \overline{n})\overline{V}_{R2}\right] \cdot \overline{n}$

$$= \rho_2 (\overline{V}_{R2} \cdot \overline{n})^2$$

Normal momentum flux below the interface $= \rho_1 (\overline{V}_{R1} \cdot \overline{n})^2$

External stress in normal direction $= \sigma_2 - \sigma_1 - \frac{\Gamma}{R}$

Hence,

$$\rho_2 (\overline{V}_{R2} \cdot \overline{n})^2 - \rho_1 (\overline{V}_{R1} \cdot \overline{n})^2 = \sigma_2 - \sigma_1 - \frac{\Gamma}{R}$$

Following the development of Eq. (3.4.16) it is seen that

$$\overline{V}_{R2} \cdot \overline{n} = \frac{1}{|\nabla F|} \frac{D_2 F}{D_2 t} \text{ and } \overline{V}_{R1} \cdot \overline{n} = \frac{1}{|\nabla F|} \frac{D_1 F}{D_1 t}$$

Therefore,

$$\frac{1}{|\nabla F|^2} \left[\rho_2 \left(\frac{D_2 F}{D_2 t} \right)^2 - \rho_1 \left(\frac{D_1 F}{D_1 t} \right)^2 \right] = \sigma_2 - \sigma_1 - \frac{\Gamma}{R} \tag{A}$$

For $F(x,y,t) = y - \eta(x,t)$

$$\frac{1}{R} = - \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}$$

and

$$|\nabla F| = \left[\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 \right]^{1/2} = \left(1 + \eta_x^2 \right)^{1/2}$$

Substituting for $1/R$, $|\nabla F|$ and $\sigma_{1,2}$ (from Appendix B) into Eq. (A) and multiplying through by $(1 + \eta_x^2)^{1/2}$ the following equation is obtained

$$\begin{aligned} \rho_2 \left(\frac{D_2 F}{D_2 t} \right)^2 - \rho_1 \left(\frac{D_1 F}{D_1 t} \right)^2 = {} & \frac{\Gamma \eta_{xx}}{1 + \eta_x^2} - (\overline{p}_2 - \overline{p}_1)\,(1 + \eta_x^2) \\ & + 2\eta_x^2 \left(\mu_2 \frac{\partial \overline{u}_2}{\partial x} - \mu_1 \frac{\partial \overline{u}_1}{\partial x} \right) \\ & + 2 \left(\mu_2 \frac{\partial \overline{v}_2}{\partial y} - \mu_1 \frac{\partial \overline{v}_1}{\partial y} \right) \\ & - 2\eta_x \left[\mu_2 \left(\frac{\partial \overline{u}_2}{\partial y} + \frac{\partial \overline{v}_2}{\partial x} \right) - \mu_1 \left(\frac{\partial \overline{u}_1}{\partial y} + \frac{\partial \overline{v}_1}{\partial x} \right) \right] \end{aligned}$$

$$\text{at } y = \eta(x,t) \tag{3.4.19}$$

In the absence of mass transfer Eq. (3.4.19) reduces to Eq. (2.4.10) and for steady-state conditions it reduces to (3.2.19).

11. Specified injection velocity at the wall

$$\bar{v}_1(x,y,t) = \dot{m}/\rho_1, \quad y = -h \tag{3.4.20}$$

where $\dot{m}$ is known from the steady-state solution of the Clausius-Clayperon equation.

12. Global mass balance at the interface (Fig. 2)

This condition can be expressed as

Mass flux below the interface
= Mass flux above the interface

i.e.

$$\rho_1 \bar{V}_{R1} \cdot \bar{n} = \rho_2 \bar{V}_{R2} \cdot \bar{n}$$

which reduces to

$$\rho_2 \frac{D_2 F}{D_2 t} = \rho_1 \frac{D_1 F}{D_1 t} \quad \text{at } y = \eta(x,t) \tag{3.4.21}$$

as shown in the derivation of Eq. (3.4.16)

13. Balance of liquid species across the interface

This is the condition that the mass flux of liquid at the interface is balanced by the convection and diffusion of vapor species away from the interface. Thus

$$\rho_1 \bar{V}_{R1} \cdot \bar{n} = \rho_2 \bar{X} \bar{V}_{R2} \cdot \bar{n} - \rho_2 D \frac{\partial \bar{X}}{\partial n}$$

This equation reduces to Eq. (3.2.22) in the steady-state case.

Carrying out the usual substitutions

$$\rho_1 \frac{1}{|\nabla F|} \frac{D_1 F}{D_1 t} = \rho_2 \bar{\chi} \frac{1}{|\nabla F|} \frac{D_2 F}{D_2 t} - \rho_2 D \frac{1}{|\nabla F|} \nabla \bar{\chi} \cdot \nabla F$$

or

$$\rho_1 \frac{D_1 F}{D_1 t} = \rho_2 \bar{\chi} \frac{D_2 F}{D_2 t} - \rho_2 D \nabla \bar{\chi} \cdot \nabla F \quad \text{at } y = \eta(x,t)$$

combining with Eq. (3.4.21) the final form is

$$(1 - \bar{\chi}) \frac{D_1 F}{D_1 t} + D \nabla \bar{\chi} \cdot \nabla F = 0 \quad \text{at } y = \eta(x,t) \qquad (3.4.22)$$

Since it has been assumed that $Pr_2 = Le_2 = 1$ in the unsteady case also, $D = \nu_2$ in Eq. (3.4.22).

14. Edge condition on vapor concentration

$$\bar{\chi}(x,y,t) = 0 \quad \text{at } y = \delta \qquad (3.4.23)$$

The boundary conditions (3.4.10) through (3.4.23) developed so far are based on the same physical conditions as in the steady-state case. The order of the steady-state system of governing equations (3.2.1) - (3.2.9) is 14, whereas the order of the unsteady system (3.4.1) - (3.4.9) is 16 with respect to the variable y. As pointed out in Sec. 2.4, the x and t dependence may be eliminated by assuming a travelling wave type of unsteadiness and consequently one need only be concerned with boundary conditions with respect to y. The change in the order from 14 to 16 is due to the appearance of second derivatives w.r.t. y in Eqs. (3.4.3) and (3.4.7). Therefore two additional boundary conditions (on $\overline{v}_1$ and $\overline{v}_2$) are required for a well-posed formulation. One straight-forward condition, analogous to Eq. (2.4.12) in the zero mass transfer case, is

15. $$\overline{v}_2(x,y,t) = \dot{m}/\rho_2, \; y = \delta \tag{3.4.24}$$

where $\dot{m}$ is the steady-state value. Thus the mass flux leaving the edge of the boundary layer is assumed to remain unchanged.

The last remaining condition is not so obvious. It expresses the fact that Eq. (3.4.21) can in fact be looked upon as two boundary conditions, viz.

12a. $$\rho_1 \frac{D_1 F}{D_1 t} = \overline{\dot{m}}(x,y,t) \quad \text{at } y = \eta(x,t) \tag{3.4.21a}$$

12b. $$\rho_2 \frac{D_2 F}{D_2 t} = \overline{\dot{m}}(x,y,t) \quad \text{at } y = \eta(x,t) \tag{3.4.21b}$$

where $\overline{\dot{m}}$ is the unsteady mass flux at the interface and its determination will be discussed later in this section. Eqs. (3.4.1) - (3.4.9) together with the boundary conditions (3.4.10) - (3.4.20), (3.4.21a), (3.4.21b) and (3.4.22) - (3.4.24) form a well-posed problem.

In the steady-state problem the mass transfer rate $\dot{m}$ was obtained by applying the condition of phase equilibrium at the interface. It is now assumed that the equilibrium of phases prevails in the unsteady case also. This requires that the Clausius-Clayperon equation be satisfied in the unsteady case. Writing Eq. (3.3.3) for the unsteady problem

16. $$\overline{Xp_2} = Ke^{-\ell/R\overline{T}_2} \quad \text{at } y = \eta(x,t) \qquad (3.4.25)$$

The Clausius-Clayperon equation thus determines the steady (or mean) mass transfer rate $\dot{m}$ for the steady-state problem and the perturbed mass transfer rate $\bar{\dot{m}}$ in the unsteady case. With the aid of Eq. (3.4.25) the previously stated formulation can be modified as follows.

A well-posed unsteady formulation is represented by governing Eqs. (3.4.1) - (3.4.9) and the boundary conditions (3.4.10) - (3.4.25). It should be noted that this does not require the use of Eqs. (3.4.21a) and (3.4.21b).

3.5 Travelling Wave Solution of the Unsteady Problem

The solution of the unsteady problem closely follows the procedures in Secs. 2.5 and 2.6. The experience gained in the solution of the unsteady zero mass transfer problem suggests the following travelling wave solution.

$$q_i(x,y,t) = q_i(y) + sq_i(y)e^{i(kx-\omega t)} + O(s^2) \quad \underset{i=1,2}{} \qquad (3.5.1)$$

where $s \ll 1$

The possibility of assuming the above form of solution was mentioned earlier in connection with Eq. (2.6.8). The interface shape is assumed to be

$$\eta(x,t) = she^{i(kx-\omega t)} + 0(s^2) \tag{3.5.2}$$

The transfer of boundary conditions from the unknown interface $y = \eta(x,t)$ to the known steady-state position $y = 0$ is accomplished by the Taylor series expansion

$$\overline{q}_i(x,\eta,t) = \overline{q}_i(x,0,t) + \left.\frac{\partial \overline{q}}{\partial y}\right|_{y=0} \eta + \left.\frac{\partial^2 \overline{q}}{\partial y^2}\right|_{y=0} \frac{\eta^2}{2!} + ---, \quad i=1,2 \tag{3.5.3}$$

where η is given by (3.5.2)

Substituting Eqs. (3.5.1) - (3.5.3) into the governing Eqs. (3.4.1) - (3.4.9) and the boundary conditions (3.4.10) - (3.4.25) and subtracting the zeroth order (i.e. steady-state) equations, the 0(s) problem is (after considerable algebra):

Liquid:

1. $$iku_1 + v_1' = 0 \tag{3.5.4}$$

2. $$-i\omega u_1 + iku_1\tilde{u}_1 + v_1\tilde{u}_1' + \tilde{v}_1 u_1' = -\frac{ikp_1}{\rho_1} + \nu_1(u_1'' - k^2 u_1) \tag{3.5.5}$$

3. $$-i\omega v_1 + ikv_1\tilde{u}_1 + \tilde{v}_1 v_1' = -\frac{p_1'}{\rho_1} + \nu_1(v_1'' - k^2 v_1) \tag{3.5.6}$$

4. $$-i\omega T_1 + ikT_1\tilde{u}_1 + v_1\tilde{T}_1' + \tilde{v}_1 T_1' = \frac{k_1}{\rho_1 c_{p1}}(T_1'' - k^2 T_1) \tag{3.5.7}$$

Gas:

5. $$iku_2 + v_2' = 0 \tag{3.5.8}$$

6. $$-i\omega u_2 + ik\tilde{u}_2 u_2 + v_2\tilde{u}_2' + \tilde{v}_2 u_2' = -\frac{ikp_2}{\rho_2} + \nu_1(u_2'' - k^2 u_2) \qquad (3.5.9)$$

7. $$-i\omega v_2 + ikv_2\tilde{u}_2 + \tilde{v}_2 v_2' = -\frac{p_2'}{\rho_2} + \nu_2(v_2'' - k^2 v_2) \qquad (3.5.10)$$

8. $$-i\omega T_2 + ikT_2\tilde{u}_2 + v_2\tilde{T}_2' + \tilde{v}_2 T_2' = \frac{k_2}{\rho_2 c_{p2}}(T_2'' - k^2 T_2) \qquad (3.5.11)$$

9. $$-i\omega\chi + ik\chi\tilde{u}_2 + v_2\tilde{\chi}' + \tilde{v}_2\chi' = D(\chi'' - k^2\chi) \qquad (3.5.12)$$

Boundary conditions are

1. $$u_1(y) = 0, \; y = -h \qquad (3.5.13)$$

2. $$u_2(y) = 0, \; y = \delta \qquad (3.5.14)$$

3. $$u_2 - u_1 = h(\tilde{u}_1' - \tilde{u}_2') + ikh(\tilde{v}_1 - \tilde{v}_2), \; y = 0 \qquad (3.5.15)$$

4. $$\mu_2(u_2' + ikv_2) - \mu_1(u_1' + ikv_1) = (\mu_1\tilde{u}_1'' - \mu_2\tilde{u}_2'')h, \; y = 0 \qquad (3.5.16)$$

5. $$T_1(y) = 0, \; y = 0$$

or $\qquad (3.5.17)$

$$T_1'(y) = 0, \; y = 0$$

6. $$T_2(y) = 0, \; y = 0 \qquad (3.5.18)$$

7. $$k_2 T_2' - k_1 T_1' = \ell\rho_2\left[v_2 - ikh(\tilde{u}_2 - w/k)\right] + (k_1\tilde{T}_1'' - k_2\tilde{T}'')h, \; y = 0 \qquad (3.5.19)$$

8. $$T_2 - T_1 = (\tilde{T}_1' - \tilde{T}_2')h, \; y = 0 \qquad (3.5.20)$$

9. $$p_2(y) = 0, \; y = \delta \qquad (3.5.21)$$

10. $$2\dot{m}\left[(v_2 - v_1) - ikh(\tilde{u}_2 - \tilde{u}_1)\right] = -\Gamma k^2 h - (p_2 - p_1) - (\tilde{p}_2' - \tilde{p}_1')h$$

$$+ 2(\mu_2 v_2' - \mu_1 v_1'), \quad y = 0 \qquad (3.5.22)$$

11. $$v_1(y) = 0, \quad y = -h \qquad (3.5.23)$$

12. $$\rho_2\left[v_2 - ikh(\tilde{u}_2 - \omega/k)\right] = \rho_1\left[v_1 - ikh(\tilde{u}_1 - \omega/k)\right], \quad y = 0 \qquad (3.5.24)$$

13. $$(1 - \tilde{\chi})\left[v_2 - ikh(\tilde{u}_2 - \omega/k)\right] - \tilde{v}_2\chi + D\chi' = 0, \quad y = 0 \qquad (3.5.25)$$

14. $$\chi(y) = 0, \quad y = \delta \qquad (3.5.26)$$

15. $$v_2(y) = 0, \quad y = \delta \qquad (3.5.27)$$

16. $$\frac{p_2}{\tilde{p}_2} + \frac{\chi}{\tilde{\chi}} = \frac{\ell T_2}{R\tilde{T}_2^2}, \quad y = 0 \qquad (3.5.28)$$

The total order of the system of ordinary differential equations (3.5.4) - (3.5.12) is 16 and there are 16 boundary conditions (3.5.13) - (3.5.28).

3.6 Further Simplifications

u_1 and p_1 may be eliminated from Eqs. (3.5.4) - (3.5.6) using the procedure of Sec. 2.7 to obtain an equation in v_1. Similarly, an equation in v_2 can be derived by combining equations (3.5.8) - (3.5.10). Now the governing equations are

1. $$v_1^{iv} - \frac{\tilde{v}_1}{\nu_1} v_1''' - 2k^2 v_1'' + \frac{k^2}{\nu_1} v_1 v_1' + k^4 v_1$$

$$= \frac{ik}{\nu_1}\left[(v_1'' - k^2 v_1)(\tilde{u}_1 - \omega/k) - \tilde{u}_1'' v_1\right] \qquad (3.6.1)$$

2. $$-i\omega T_1 + ikT_1\tilde{u}_1 + v_1\tilde{T}_1' + \tilde{v}_1T_1' = \frac{k_1}{\rho_1 c_{p1}}(T_1'' - k^2T_1) \quad (3.6.2)$$

3. $$v_2^{iv} - \frac{\tilde{v}_2}{\nu_2}v_2''' - 2k^2v_2'' + \frac{k^2}{\nu_2}\tilde{v}_2v_2' + k^4v_2$$
$$= \frac{ik}{\nu_2}\left[(v_2'' - k^2v_2)(\tilde{u}_2 - \omega/k) - u_2''v_2\right] \quad (3.6.3)$$

4. $$-i\omega T_2 + ikT_2\tilde{u}_2 + v_2\tilde{T}_2' + \tilde{v}_2T_2' = \frac{k_2}{\rho_2 c_{p2}}(T_2'' - k^2T_2) \quad (3.6.4)$$

5. $$-i\omega\chi + ik\chi\tilde{u}_2 + v_2\tilde{\chi}' + \tilde{v}_2\chi' = D(\chi'' - k^2\chi) \quad (3.6.5)$$

A comparison of Eq. (3.6.1) and the original Orr-Sommerfeld equation (2.7.1) shows that there are two additional terms in the former. These terms $\tilde{v}_1v_1'''/\nu_1$ and $k^2\tilde{v}_1v_1'/\nu_1$ are due to mass transfer. A similar observation can be made by comparing Eqs. (3.6.3) and (2.7.2)

The variables u_1, p_1, u_2 and p_2 may be eliminated from the boundary conditions using equations (3.5.4), (3.5.5), (3.5.8) and (3.5.9). This operation is the same as described in Sec. 2.7. The resulting boundary conditions are

1. $$v_1' = 0, \; y = -h \quad (3.6.6)$$

2. $$v_2' = 0, \; y = \delta \quad (3.6.7)$$

3. $$v_1' - v_2' = ikh(\tilde{u}_1' - \tilde{u}_2') - k^2h(\tilde{v}_1 - \tilde{v}_2), \; y = 0 \quad (3.6.8)$$

4. $$\mu_1(v_1'' + k^2v_1) - \mu_2(v_2'' + k^2v_2) = ikh(\mu_1\tilde{u}_1'' - \mu_2\tilde{u}_2''), \; y = 0 \quad (3.6.9)$$

5. $$T_1 = 0, \; y = -h \quad \text{or} \quad T_1' = 0, \; y = -h \quad (3.6.10)$$

6. $$T_2 = 0, \ y = \delta \tag{3.6.11}$$

7. $$k_2 T_2' - k_1 T_1' = \ell \rho_2 \left[v_2 - ikh(\tilde{u}_2 - \omega/k) \right] + (k_1 \tilde{T}_1'' - k_2 \tilde{T}_2'')h, \ y = 0 \tag{3.6.12}$$

8. $$T_2 - T_1 = (\tilde{T}_1' - \tilde{T}_2')h, \ y = 0 \tag{3.6.13}$$

9. $$\Gamma k^2 h + h(\tilde{p}_2' - \tilde{p}_1') + \frac{1}{k^2}\left[\mu_2 (v_2''' - k^2 v_2') - \mu_1 (v_1''' - k^2 v_1') + \dot{m}(v_1'' - v_2'') \right]$$
$$+ \frac{i}{k}\left[\rho_2 \{ v_2 \tilde{u}_2' - v_2'(\tilde{u}_2 - \omega/k) \} - \rho_1 \{ v_1 \tilde{u}_1' - v_1'(\tilde{u}_1 - \omega/k) \} \right]$$
$$-2(\mu_2 v_2' - \mu_1 v_1') + 2\dot{m}(v_2 - v_1) = 0, \ y = 0 \tag{3.6.14}$$

10. $$v_1 = 0, \ y = -h \tag{3.6.15}$$

11. $$\rho_2 \left[v_2 - ikh(\tilde{u}_2 - \omega/k) \right] = \rho_1 \left[v_1 - ikh(\tilde{u}_1 - \omega/k) \right], \ y = 0 \tag{3.6.16}$$

12. $$(1 - \tilde{\chi}) \left[v_2 - ikh(\tilde{u}_2 - \omega/k) \right] - \tilde{v}_2 \chi + D\chi' = 0, \ y = 0 \tag{3.6.17}$$

13. $$\chi = 0, \ y = \delta \tag{3.6.18}$$

14. $$v_2 = 0, \ y = \delta \tag{3.6.19}$$

15. $$\frac{\mu_2}{\tilde{p}_2 k^2}\left[v_2''' - k^2 v_2' - \frac{\dot{m}}{\mu_2} v_2'' \right] - \frac{\rho_2}{ik\tilde{p}_2}\left[v_2 \tilde{u}_2' - v_2(\tilde{u}_2 - \omega/k) \right] + \frac{\chi}{\tilde{\chi}}$$
$$= \frac{\ell T_2}{R\tilde{T}_2^2}, \ y = 0 \tag{3.6.20}$$

The following observations can now be made:

(i) The order of the governing equation (3.6.1) - (3.6.5) is 14 whereas the number of boundary conditions is 15. The condition (3.5.21) is no

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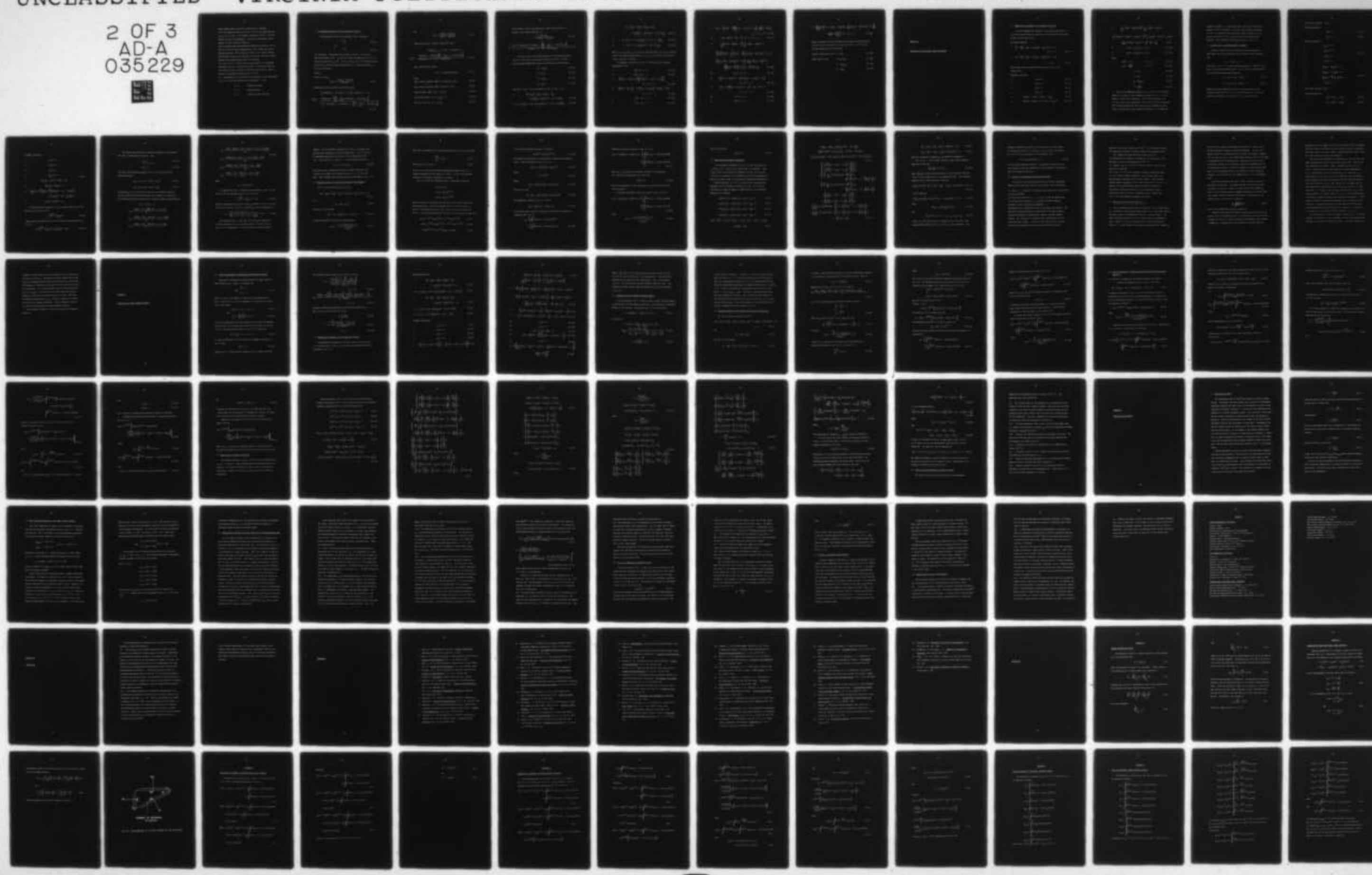

longer needed since p_2 has been eliminated as an unknown.

(ii) The governing equations (3.6.1) - (3.6.5) are homogeneous in v_1, T_1, v_2, T_2, and χ. The boundary conditions (3.6.6) - (3.6.20), however, are not all homogeneous. This fact has important consequences as will be shown in Chapter V.

(iii) It appears that the modified Orr-Sommerfeld equations (3.6.1) and (3.6.3) can be solved independently of the energy and concentration equations. These equations, however, are coupled through the boundary conditions and therefore, unlike the zero mass transfer problem, the present problem cannot be decoupled.

(iv) Eqs. (3.6.1) - (3.6.5) can be solved subject to 14 boundary conditions (3.6.6) - (3.6.15) and (3.6.17) - (3.6.20) and then Eq. (3.6.16) can be used to obtain a characteristic equation. This is similar to the method described in Sec. 2.7(iv).

(v) The stability of the interface is determined in the same manner as in Sec. 2.7(v) by solving for the eigenvalue ω. Thus

$\omega_i > 0$	unstable interface
$\omega_i < 0$	stable interface
$\omega_i = 0$	neutrally stable interface

3.7 Non-dimensionalization of the Eigenvalue Problem

Non-dimensional vertical co-ordinates ξ and η defined by

$$\xi = \frac{y}{h} \tag{3.7.1}$$

and

$$\eta = \frac{y}{\delta} \tag{3.7.2}$$

are introduced. The steady-state profiles (3.2.26) - (3.2.34) are made dimensionless first. $\tilde{u}_1$ and $\tilde{T}_1$ are made non-dimensional w.r.t. the interface quantities u_{if} (Eq. 3.2.36) and T_{if} (Eq. 3.2.37) respectively. Similarly $\tilde{u}_2$ and $\tilde{T}_2$ are made dimensionless w.r.t. edge conditions u_e and T_e respectively.

Liquid:

u-velocity profile

$$\hat{u}_1(\xi) = \frac{\exp(R_h\xi) - \exp(-R_h)}{1 - \exp(-R_h)} \tag{3.7.3}$$

Temperature profile (constant temperature wall)

$$\hat{T}_1(\xi) = \frac{\begin{array}{c}\bar{c}_p\{\exp(R_hPr_1\xi) - \exp(-R_hPr_1)\} - \bar{c}_p \frac{T_w}{T_e}\{\exp(R_hPr_1\xi) - 1\} + \\ \{\exp(R_\delta Pr_2) - 1\}\left[\frac{T_w}{T_e} - \frac{\ell}{c_{p1}T_e}\{\exp(R_hPr_1\xi) - \exp(-R_hPr_1)\}\right]\end{array}}{\bar{c}_p\{1 - \exp(-R_hPr_1)\} + \{\exp(R_\delta Pr_2) - 1\}\left[\frac{T_w}{T_e} - \frac{\ell}{c_{p1}T_e}\{1 - \exp(-R_hPr_1)\}\right]} \tag{3.7.4}$$

Gas:

$$\hat{u}_2(\eta) = \frac{\exp(R_\delta \eta) - \exp(-R_h)}{\exp(R_\delta) - \exp(-R_h)} \tag{3.7.5}$$

Temperature profile (constant temperature wall)

$$\hat{T}_2(\eta) = \frac{\begin{gathered}\exp(R_\delta Pr_2 \eta - 1) + \bar{c}_p\{1 - \exp(-R_h Pr_1)\} + \\ \{\exp(R_\delta Pr_2) - \exp(R_\delta \eta)\}\left[\frac{T_w}{T_e} - \frac{\ell}{c_{p1}T_e}\{1 - \exp(-R_h Pr_1)\}\right]\end{gathered}}{\{\exp(R_\delta Pr_2) - 1\} + \bar{c}_p\{1 - \exp(-R_h Pr_1)\}} \tag{3.7.6}$$

Vapor mass fraction profile

$$\hat{\chi}(\eta) = 1 - \exp(R_\delta \eta)/\exp(R_\delta) \tag{3.7.7}$$

where

Mass transfer Reynolds number for liquid $R_h = \dot{m}h/\mu_1$ (3.7.8)

Mass transfer Reynolds number for gas $R_\delta = \dot{m}\delta/\mu_1$ (3.7.9)

Liquid Prandtl number $Pr_1 = \mu_1 c_{p1}/k_1$ (3.7.10)

Gas Prandtl number $Pr_2 = \mu_2 c_{p2}/k_2 = 1$ (3.7.11)

Specific heat ratio $\bar{c}_p = c_{p2}/c_{p1}$ (3.7.12)

The interface velocity and temperature, made dimensionless w.r.t. boundary layer edge properties, are

$$\bar{u} = \frac{1 - \exp(-R_h)}{\exp(R_\delta) - \exp(-R_h)} \tag{3.7.13}$$

$$\bar{T} = \frac{\bar{c}_p\{1 - \exp(-R_h Pr_1)\} + \{\exp(R_\delta) - 1\}\left[\frac{T_w}{T_e} - \frac{\ell}{c_{p1} T_e}\{1 - \exp(-R_h Pr_1)\}\right]}{\{\exp(R_\delta) - 1\} + \bar{c}_p\{1 - \exp(-R_h Pr_1)\}} \tag{3.7.14}$$

The next step is to non-dimensionalize the governing equations (3.6.1) - (3.6.5) and the boundary conditions (3.6.6) - (3.6.20). The following dimensionless terms are introduced for this purpose.

$$\psi_1 = v_1/u_{if} \tag{3.7.15}$$

$$\psi_2 = v_2/u_e \tag{3.7.16}$$

$$\theta_1 = T_1/T_{if} \tag{3.7.17}$$

$$\theta_2 = T_2/T_e \tag{3.7.18}$$

Then Eqs. (3.6.1) - (3.6.5) assume the form, for $Pr_2 = Le_2 = 1$

1. $$\psi_1^{iv} - R_h\psi_1''' - 2\alpha_1^2\psi_1'' + \alpha_1^2 R_h \psi_1' + \alpha_1^4\psi_1 = i\alpha_1 R_1\left[(\psi_1'' - \alpha_1^2\psi_1)(\hat{u}_1(\xi) - c_1) - \hat{u}_1''\psi_1\right] \tag{3.7.19}$$

2. $$\theta_1'' - Pr_1 R_h \theta_1' - \{\alpha_1^2 + i\alpha_1 Pr_1 R_1(\hat{u}_1(\xi) - c_1)\}\theta_1 = Pr_1 R_1 \hat{T}_1' \psi_1 \tag{3.7.20}$$

3. $$\ddddot{\psi}_2 - R_\delta \dddot{\psi}_2 - 2\alpha_2^2 \ddot{\psi}_2 + \alpha_2^2 R_\delta \dot{\psi}_2 + \alpha_2^4 \psi_2$$

$$= i\alpha_2 R_2 \left[(\ddot{\psi}_2 - \alpha_2^2 \psi_2)(\hat{u}_2(\eta) - c_2) - \ddot{\hat{u}}_2 \psi_2 \right] \qquad (3.7.21)$$

4. $$\ddot{\theta}_2 - R_\delta \dot{\theta}_2 - \{\alpha^2 + i\alpha_2 R_2 (\hat{u}_2(\eta) - c_2)\}\, \theta_2 = R_2 \dot{\hat{T}}_2 \psi_2 \qquad (3.7.22)$$

5. $$\ddot{\chi} - R_\delta \dot{\chi} - \{\alpha^2 + i\alpha_2 R_2 (\hat{u}_2(\eta) - c_2)\}\chi = R_2 \dot{\hat{\chi}} \psi_2 \qquad (3.7.23)$$

where α_1, α_2, R_1, R_2, c_1, c_2 have the same definitions as in Eqs. (2.8.12) - (2.8.17) and relationships between α_1, α_2, c_1, c_2 and R_1, R_2 are given by Eqs. (2.8.18) - (2.8.20).

The boundary conditions (3.6.6) - (3.6.20) take the following dimensionless form

1. $$\psi_1'(\xi) = 0, \quad \xi = -1 \qquad (3.7.24)$$

2. $$\dot{\psi}_2(\eta) = 0, \quad \eta = 1 \qquad (3.7.25)$$

3. $$\overline{u}\left[\psi_1'(\xi) - i\alpha_1 \hat{u}_1' + \alpha_1^2 \frac{R_h}{R_1}\right] = \varepsilon \left[\dot{\psi}_2(\eta) - i\alpha_1 \dot{\hat{u}}_2 + \alpha_1 \alpha_2 \frac{R_\delta}{R_2}\right]$$

$$\text{at } \xi = 0, \ \eta = 0 \qquad (3.7.26)$$

4. $$\overline{u}\left[\psi_1''(\xi) + \alpha_1^2 \psi_1(\xi)\right] = \overline{\mu}\varepsilon^2 \left[\ddot{\psi}_2(\eta) + \alpha_2^2 \psi_2(\eta)\right] - i\alpha_1 (\overline{u}\hat{u}_1'' - \overline{\mu}\varepsilon^2 \ddot{\hat{u}}_2)$$

$$\text{at } \xi = 0, \ \eta = 0 \qquad (3.7.27)$$

5. $$\theta_1(\xi) = 0, \quad \xi = -1 \qquad (3.7.28)$$

6. $$\theta_1(\eta) = 0, \quad \eta = 1 \qquad (3.7.29)$$

7. $$\dot{\theta}_2(\eta) - \frac{\overline{T}}{\varepsilon\overline{k}}\,\theta_1'(\xi) = \frac{R_2\overline{T}\Lambda}{\overline{c}_p}\left[\psi_2(\eta) - i\alpha_1(\hat{u}_2(\eta) - c_2)\right] + \frac{1}{\varepsilon\overline{k}}\left[\overline{T}\hat{T}_1'' - \overline{k}\varepsilon^2\ddot{\hat{T}}_2\right]$$

at $\xi = 0,\ \eta = 0$ (3.7.30)

8. $$\theta_2(\eta) - \overline{T}\theta_1(\xi) = \overline{T}\hat{T}_1' - \varepsilon\dot{\hat{T}}_2 \quad \text{at } \xi = 0,\ \eta = 0 \qquad (3.7.31)$$

9. $$\frac{1}{\alpha_2^2R_2}\{\dddot{\psi}_2(\eta) - \alpha_2^2\dot{\psi}_2(\eta)\} - \frac{\overline{u}^2}{\overline{\rho}}\frac{1}{\alpha_1^2R_1}\{\psi_1'''(\xi) - \alpha_1^2\psi_1'(\xi)\} +$$

$$\frac{\overline{u}^2}{\overline{\rho}}\frac{1}{\alpha_1^2}\frac{R_h}{R_1}\psi_1''(\xi) - \frac{1}{\alpha_2^2}\frac{R_\delta}{R_2}\ddot{\psi}_2(\eta) + i\left[\frac{1}{\alpha_2}\{\psi_2(\eta)\dot{\hat{u}}_2 - \dot{\psi}_2(\eta)(\hat{u}_2(\eta) - c_2)\} -\right.$$

$$\left.\frac{\overline{u}^2}{\overline{\rho}}\frac{1}{\alpha_1}\{\psi_1(\xi)\hat{u}_1' - \psi_1'(\xi)(\hat{u}_1(\xi) - c_1)\}\right] - \frac{2}{\varepsilon\overline{\mu}R_2}\{\varepsilon\overline{\mu}\dot{\psi}_2(\eta) - \overline{u}\psi_1'(\xi)\} +$$

$$2\left[\frac{R_\delta}{R_2}\psi_2(\eta) - \frac{\overline{u}^2}{\overline{\rho}}\frac{R_h}{R_1}\psi_1(\xi)\right] = -\frac{\overline{u}^2}{\overline{\rho}}\left(\alpha_1^2W^2 + \frac{1}{F^2}\right) \qquad (3.7.32)$$

10. $$\psi_1(\xi) = 0,\ \xi = -1 \qquad (3.7.33)$$

11. $$\overline{\rho}\left[\psi_2(\eta) - i\alpha_1(\hat{u}_2(\eta) - c_2)\right] = \overline{u}\left[\psi_1(\xi) - i\alpha_1(\hat{u}_1(\xi) - c_1)\right]$$

at $\xi = 0,\ \eta = 0$ (3.7.34)

12. $$R_2(1 - \hat{\chi}(\eta))\left[\psi_2(\eta) - i\alpha_1(\hat{u}_2(\eta) - c_2)\right] - R_\delta\chi + \dot{\chi}(\eta) = 0$$

at $\xi = 0,\ \eta = 0$ (3.7.35)

13. $$\chi(\eta) = 0,\ \eta = 1 \qquad (3.7.36)$$

14. $$\psi_2(\eta) = 0,\ \eta = 1 \qquad (3.7.37)$$

$$15.\quad \frac{1}{E}\left[\frac{1}{\alpha_2^2 R_2}\left\{\dddot{\psi}_2(\eta) - R_\delta \ddot{\psi}_2(\eta) - \alpha_2^2 \dot{\psi}_2(\eta)\right\} - \frac{1}{i\alpha_2}\left\{\dot{\hat{u}}_2 \psi_2(\eta) - \dot{\psi}_2(\eta)(\hat{u}_2(\eta) - c_2)\right\}\right]$$

$$+ \frac{\chi(\eta)}{\hat{\chi}(\eta)} = \bar{\bar{R}}\bar{T}\frac{\theta_2(\eta)}{\hat{T}_2^2(\eta)} \text{ at } \xi = 0,\ \eta = 0 \qquad (3.7.38)$$

where in addition to the non-dimensional parameters defined by Eqs. (2.8.6), (2.8.7), (2.8.30), (2.8.31), and (3.7.8) - (3.7.14) the following quantities are introduced.

thermal conductivity ratio $\bar{k} = k_2/k_1$ (3.7.39)

Euler number of gas $E = p_e/\rho_2 u_e^2$ (3.7.40)

$\Lambda = \ell/c_{p1}T_{if}$ (3.7.41)

$\bar{\bar{R}} = \ell/RT_{if}$ (3.7.42)

CHAPTER IV

SOLUTION OF THE ZERO MASS TRANSFER PROBLEM

4.1 Mathematical Statement of the Eigenvalue Problem

A concise mathematical statement of the problem described in Sec. 2.8 is given below for linear mean velocity profiles described by Eqs. (2.8.3) and (2.8.4).

Governing equations:

1. $$\psi_1^{iv} - 2\alpha_1^2\psi_1'' + \alpha_1^4\psi_1 = i\alpha_1 R_1(\psi_1'' - \alpha_1^2\psi_1)(\hat{u}_1(\xi) - c_1)$$
$$-1 \leq \xi < 0 \qquad (4.1.1)$$

2. $$\ddot{\ddot{\psi}}_2 - 2\alpha_2^2\ddot{\psi}_2 + \alpha_2^4\psi_2 = i\alpha_2 R_2(\ddot{\psi}_2 - \alpha_2^2\psi_2)(\hat{u}_2(\eta) - c_2)$$
$$0 < \eta \leq 1 \qquad (4.1.2)$$

where primes and dots denote differentiation w.r.t. ξ *and* η respectively.

Boundary conditions:

1. $$\psi_1(-1) = 0 \qquad (4.1.3)$$

2. $$\psi_1'(-1) = 0 \qquad (4.1.4)$$

3. $$\psi_2(1) = 0 \qquad (4.1.5)$$

4. $$\dot{\psi}_2(1) = 0 \qquad (4.1.6)$$

5. $$\overline{u}\{\psi_1'(0) - i\alpha_1\hat{u}_1'\} = \varepsilon\{\dot{\psi}_2(0) - i\alpha_1\dot{\hat{u}}_2\} \qquad (4.1.7)$$

6. $$\overline{u}\{\psi_1''(0) + \alpha_1^2\psi_1(0)\} = \overline{\mu}\varepsilon^2\{\ddot{\psi}_2(0) + \alpha_2^2\psi_2(0)\} \qquad (4.1.8)$$

7. $$\frac{1}{\alpha_2^2 R_2}\{\dddot{\psi}_2(0) - \alpha_2^2\dot{\psi}_2(0)\} - \frac{\overline{u}^2}{\overline{\overline{\rho}}}\frac{1}{\alpha_1^2 R_1}\{\psi_1'''(0) - \alpha_1^2\psi_1'(0)\}$$

$$+ \frac{i}{\alpha_2}\{\dot{\hat{u}}_2\psi_2(0) - (\hat{u}_2(0) - c_2)\dot{\psi}_2(0)\} - \frac{\overline{u}^2}{\rho}\frac{i}{\alpha_1}\{\hat{u}_1'\psi_1(0) - (\hat{u}_1(0) - c_1)\psi_1'(0)\}$$

$$- \frac{2}{\varepsilon\overline{\overline{\mu}}R_2}\{\varepsilon\overline{\mu}\dot{\psi}_2(0) - \overline{u}\psi_1'(0)\} = - \frac{\overline{u}^2}{\overline{\overline{\rho}}}(\alpha_1^2 W^2 + \frac{1}{F^2}) \tag{4.1.9}$$

8. $$\psi_1(0) - i\alpha_1(\hat{u}_1(0) - c_1) = 0 \tag{4.1.10}$$

9. $$\psi_2(0) - i\alpha_1(\hat{u}_2(0) - c_2) = 0 \tag{4.1.11}$$

where

$$\hat{u}_1(\xi) = 1 + \xi \qquad -1 \leq \xi \leq 0 \tag{4.1.12}$$

$$\hat{u}_2(\eta) = \frac{\varepsilon\overline{\mu}+\eta}{\varepsilon\overline{\overline{\mu}}+1} \qquad 0 \leq \eta \leq 1 \tag{4.1.13}$$

$$\alpha_1 = \varepsilon\alpha_2 \tag{4.1.14}$$

$$c_2 = \overline{u}c_1 \tag{4.1.15}$$

$$R_1 = \overline{u}\varepsilon\overline{\mu}R_2/\overline{\rho} \tag{4.1.16}$$

$$\overline{u} = \frac{\varepsilon\overline{\mu}}{1+\varepsilon\overline{\overline{\mu}}} \tag{4.1.17}$$

The two Orr-Sommerfeld equations (4.1.1) and (4.1.2) are homogeneous in ψ_1 and ψ_2. The boundary conditions (4.1.3) - (4.1.11), however, are not all homogeneous. One of these conditions, say (4.1.11), can be used to make Eqs. (4.1.9) and (4.1.10) homogeneous. The resulting system will then consist of two homogeneous fourth order equations with eight boundary conditions, i.e. a legitimate eigenvalue problem. In the present work, however, the approach of Bordner et.al[37] is followed. The first eight boundary conditions (4.1.3) - (4.1.10) are used to determine eight constants of integration and then (4.1.11) is used to obtain the characteristic equation. This treatment was discussed earlier in Sec. 2.7.

4.2 Solution for a Long Wavelength Disturbance

Consider a disturbance on the interface whose wavelength is much larger than the liquid depth and the boundary layer thickness. Thus

$$\alpha_1 << 1 \text{ and } \alpha_2 << 1$$

with $\alpha_1/\alpha_2 = h/\delta = \varepsilon$. In most problems of interest ε itself is very small so α_1 must be extremely small. Let ψ_1 and ψ_2 be represented by the following straightforward expansions

$$\psi_1 = \psi_{10} + \alpha_1\psi_{11} + \alpha_1^2\psi_{12} + --- \qquad (4.2.1)$$

$$\psi_2 = \psi_{20} + \alpha_2\psi_{21} + \alpha_2^2\psi_{22} + --- \qquad (4.2.2)$$

Substituting these expansions into the governing equations and boundary conditions (4.1.1) - (4.1.10), making use of (4.1.14) and equating coefficients of equal powers of α_1, the results are

Zeroth order problem: $0(\alpha_1^0)$

Governing equations

$$\psi_{10}^{iv} = 0 \tag{4.2.3}$$

$$\ddddot{\psi}_{20} = 0 \tag{4.2.4}$$

Boundary conditions

1. $\psi_{10}(-1) = 0$

2. $\psi_{10}'(-1) = 0$

3. $\psi_{20}(1) = 0$

4. $\dot{\psi}_{20}(1) = 0$

(4.2.5)

5. $\overline{u}\psi_{10}'(0) - \varepsilon\dot{\psi}_{20}(0) = 0$

6. $\overline{u}\psi_{10}''(0) - \overline{\mu}\varepsilon^2\ddot{\psi}_{20}(0) = 0$

7. $\frac{\varepsilon^2}{R_2}\dddot{\psi}_{20}(0) - \frac{\overline{u}^2}{\overline{\rho}}\frac{1}{R_1}\psi_{10}'''(0) = 0$

8. $\psi_{10}(0) = 0$

(4.2.6)

First order problem: $0(\alpha_1)$

Governing equations

$$\psi_{11}^{iv} = iR_1(\hat{u}_1 - c_1)\psi_{10}'' \tag{4.2.7}$$

$$\ddddot{\psi}_{21} = iR_2(\hat{u}_2 - c_2)\ddot{\psi}_{20} \tag{4.2.8}$$

Boundary conditions

$$
\begin{aligned}
&1. \quad && \psi_{11}(-1) = 0 \\
&2. \quad && \psi'_{11}(-1) = 0 \\
&3. \quad && \psi_{21}(1) = 0 \\
&4. \quad && \dot{\psi}_{21}(1) = 0 \\
&5. \quad && \bar{u}\psi'_{11}(0) - \dot{\psi}_{21}(0) = i(\bar{u}\hat{u}'_1 - \varepsilon\dot{\hat{u}}_2) \\
&6. \quad && \bar{u}\psi''_{11}(0) - \bar{\mu}\varepsilon\ddot{\psi}_{21}(0) = 0 \\
&7. \quad && \frac{\varepsilon}{R_2}\dddot{\psi}_{21}(0) - \frac{\bar{u}^2}{\bar{\rho}}\frac{1}{R_1}\psi'''_{11} = \frac{i\bar{u}^2}{\bar{\rho}}\{\hat{u}'_1\psi_{10}(0) - (\hat{u}_1 - c_1)\psi'_{10}(0)\} \\
& && \qquad - i\varepsilon\{\dot{\hat{u}}_2\psi_{20}(0) - (\hat{u}_2 - c_2)\dot{\psi}_{20}(0)\} \\
&8. \quad && \psi_{11}(0) = i(\hat{u}_1(0) - c_1)
\end{aligned}
\tag{4.2.9}
$$

The characteristic equation for c_1 is obtained from Eq. (4.1.11) with the aid of Eq. (4.1.15)

$$c_1 = \frac{\hat{u}_2(0)}{\bar{\bar{u}}} + \frac{i}{\bar{\bar{u}}\alpha_1}\psi_2(0) \tag{4.2.10}$$

Substituting the expansion (4.2.2) into the above equation and using (4.1.14)

$$c_1 = \frac{\hat{u}_2(0)}{\bar{\bar{u}}} + \frac{i}{\bar{\bar{u}}\alpha_1}\psi_{20}(0) + \frac{i}{\varepsilon\bar{\bar{u}}}\psi_{21}(0) + O(\alpha_1) \tag{4.2.11}$$

The zeroth order problem is completely homogeneous and therefore has only a trivial general solution. Thus

$$\psi_{10} \equiv 0 \tag{4.2.12}$$

$$\psi_{20} \equiv 0 \tag{4.2.13}$$

The first order governing equations (4.2.7) and (4.2.8) have the general solutions

$$\psi_{11} = A_{11} + A_{12}\xi + A_{13}\xi^2 + A_{14}\xi^3 \tag{4.2.14}$$

$$\psi_{21} = A_{21} + A_{22}\eta + A_{23}\eta^2 + A_{24}\eta^3 \tag{4.2.15}$$

Introducing Eqs. (4.2.14) and (4.2.15) into the boundary conditions (4.2.9), using (4.1.16) and solving for the constants of integration, the following results are obtained after lengthy algebraic manipulations,

$$A_{11} = i(u_1(0) - c_1)$$

$$A_{12} = \frac{i\left\{(\overline{u}\hat{u}_1' - \varepsilon\dot{\hat{u}}_2) + \frac{6\overline{u}}{\varepsilon\overline{\mu}}(\hat{u}_1(0) - c_1)(1 + \frac{1}{\varepsilon})\right\}}{\overline{u}D}$$

$$A_{13} = \frac{i\left\{2(\overline{u}\hat{u}_1' - \varepsilon\dot{\hat{u}}_2) - 3\overline{u}(\hat{u}_1(0) - c_1)(1 - \frac{1}{\varepsilon^2\overline{\mu}})\right\}}{\overline{u}D}$$

$$A_{14} = \frac{i\left\{(\overline{u}\hat{u}_1' - \varepsilon\dot{\hat{u}}_2) - 2\overline{u}(\hat{u}_1(0) - c_1)(1 + \frac{1}{\varepsilon\overline{\mu}})\right\}}{\overline{u}D}$$

$$A_{21} = \frac{i\left\{2(1 + \frac{1}{\varepsilon})(\bar{u}\hat{u}_1' - \varepsilon\dot{\hat{u}}_2) - \bar{u}(\hat{u}_1(0) - c_1)(3 + \frac{4}{\varepsilon} + \frac{1}{\varepsilon^2\bar{\mu}})\right\}}{\varepsilon\bar{\mu}D}$$

$$A_{22} = \frac{i\left\{6\bar{u}(\hat{u}_1(0) - c_1)(\frac{1}{\varepsilon} - 1) - (4 + \frac{3}{\varepsilon})(\bar{u}\hat{u}_1' - \varepsilon\dot{\hat{u}}_2)\right\}}{\varepsilon\bar{\mu}D} \tag{4.2.16}$$

$$A_{23} = \frac{i\left\{2(\bar{u}\hat{u}_1' - \varepsilon\dot{\hat{u}}_2) - 3\bar{u}(\hat{u}_1(0) - c_1)(1 - \frac{1}{\varepsilon^2\bar{\mu}})\right\}}{\varepsilon\bar{\mu}D}$$

$$A_{24} = \frac{i\left\{(\bar{u}\hat{u}_1' - \varepsilon\dot{\hat{u}}_2) - 2\bar{u}(\hat{u}_1(0) - c_1)(1 + \frac{1}{\varepsilon\bar{\mu}})\right\}}{\varepsilon^2\bar{\mu}D}$$

where

$$D = 1 + \frac{1}{\varepsilon\bar{\mu}}(4 + \frac{3}{\varepsilon})$$

To complete the task of obtaining the eigenvalue c_1 Eqs. (4.2.13) and (4.2.15) are combined with Eq. (4.2.11) to get

$$c_1 = \frac{\hat{u}_2(0)}{\bar{u}} + \frac{i}{\varepsilon\bar{u}} A_{21} + 0\ (\alpha_1) \tag{4.2.17}$$

Finally, substituting for $\hat{u}_2(0)$, $\bar{u}$ and A_{21} from Eqs. (4.1.13), (4.1.17) and (4.2.16) into Eq. (4.2.17), the expression for the eigenvalue is

$$c_1 = \frac{\varepsilon^3\bar{\mu}^2 + (2\varepsilon\bar{\mu} + 1)\{\frac{1}{\varepsilon} + 2(\varepsilon + 1)\}}{\varepsilon\bar{\mu}\left[6 + 4\varepsilon(1 + \frac{1}{\varepsilon^2}) + \varepsilon^2\bar{\mu}(1 + \frac{1}{\varepsilon^4\bar{\mu}^2})\right]} + 0(\alpha_1) \tag{4.2.18}$$

The expression for c_1 shows that this particular eigenvalue (or mode) depends only on the thickness ratio ε and the viscosity ratio $\bar{\mu}$; it is independent of the film Reynolds, Froude and Weber numbers. The most important observation is that c_1 is purely real and therefore represents a neutrally stable mode. Eq. (4.2.18) can be simplified greatly for the case of a thin liquid layer with $\varepsilon, \bar{\mu} < 1$ such that $\varepsilon\bar{\mu} << 1$ and $\varepsilon^2 << 1$. This operation results in

$$c_1 = 1 + 2\varepsilon + O(\alpha_1) \tag{4.2.19}$$

This form is more illuminating in that it clearly shows that the critical point always lies inside the gas (i.e. $c_1 > 1$) for this mode. This statement will perhaps become more meaningful in Sec. 4.6.

4.3 General Solution for Arbitrary Disturbance Wave Numbers

Eq. (4.1.1) can be written in the form

$$(\psi_1'' - \alpha_1^2\psi_1)'' - \alpha_1^2(\psi_1'' - \alpha_1^2\psi_1) = i\alpha_1 R_1(\psi_1'' - \alpha_1^2\psi_1)(\hat{u}_1(\xi) - c_1) \tag{4.3.1}$$

Let

$$\psi_1'' - \alpha_1^2\psi_1 = w_1(\xi) \tag{4.3.2}$$

Then Eq. (4.3.1) becomes

$$w_1'' - \{\alpha_1^2 + i\alpha_1 R_1(\hat{u}_1(\xi) - c_1)\}w_1 = 0 \tag{4.3.3}$$

Following Feldman[28], defining the transformation,

$$\zeta_1(\xi) = -\frac{\alpha_1^2 + i\alpha_1 R_1(\hat{u}_1(\xi) - c_1)}{(\alpha_1 R_1 \hat{u}_1')^{2/3}} \tag{4.3.4}$$

where $\hat{u}_1'$ is a constant for a linear velocity profile, Eq. (4.3.3) reads

$$\frac{d^2h_1}{d\zeta_1^2} - \zeta_1 h_1 = 0 \tag{4.3.5}$$

where $h_1(\zeta_1) = w_1\{\xi(\zeta_1)\}$ (4.3.6)

Eq. (4.3.5) is the well known Airy differential equation and it is regular everywhere in the complex ζ_1 - plane. In fact, this equation possesses an irregular singularity at infinity.

Eq. (4.3.5) has the following pairs of independent solutions,

$$Ai(\zeta_1),\ Bi(\zeta_1)$$

$$Ai(\zeta_1),\ Ai(\zeta_1 e^{2\pi i/3})$$

$$Ai(\zeta_1),\ Ai(\zeta_1 e^{-2\pi i/3})$$

where Ai and Bi are called the Airy functions of the first and second kind respectively. Ai has the property that it is real for real ζ_1 and Bi is constructed from Ai in such a way that Bi is also real for real ζ_1. The relationships amongst the above solution pairs are (Ref. 44)

$$Bi(Z) = e^{\pi i/6}\ Ai(Ze^{2\pi i/3}) + e^{-\pi i/6}\ Ai(Ze^{-2\pi i/3}) \tag{4.3.7}$$

$$Ai(Z) + e^{2\pi i/3}\ Ai(Ze^{2\pi i/3}) + e^{-2\pi i/3}\ Ai(Ze^{-2\pi i/3}) \tag{4.3.8}$$

$$Ai(Ze^{\pm 2\pi i/3}) = \frac{1}{2}\ e^{\pm\pi i/3}\left\{Ai(Z) \mp iBi(Z)\right\} \tag{4.3.9}$$

In the present analysis the pair of solutions

$$\mathrm{Ai}(Ze^{2\pi i/3}),\ \mathrm{Ai}(Ze^{-2\pi i/3}) \tag{4.3.10}$$

was chosen for convenience in the numerical integration encountered later. Thus the solution to Eq. (4.3.5) is

$$h_1(\zeta_1) = C_3\mathrm{Ai}(\zeta_1 E^+) + C_4\mathrm{Ai}(\zeta_1 E^-) \tag{4.3.11}$$

where

$$E^{\pm} = e^{\pm 2\pi i/3} \tag{4.3.12}$$

Hence

$$w_1(\xi) = C_3\mathrm{Ai}\{\zeta_1(\xi)E^+\} + C_4\mathrm{Ai}\{\zeta_1(\xi)E^-\} \tag{4.3.13}$$

From Eq. (4.3.2)

$$\psi_1'' - \alpha_1^2\psi_1 = C_3\mathrm{Ai}\{\zeta_1(\xi)E^+\} + C_4\mathrm{Ai}\{\zeta_1(\xi)E^-\} \tag{4.3.14}$$

The homogeneous solution of Eq. (4.3.14) is

$$\psi_{1H} = C_1\exp(\alpha_1\xi) + C_2\exp(-\alpha_1\xi) \tag{4.3.15}$$

and the particular solution, obtained by the method of variation of parameters (Ref. 42) is

$$\begin{aligned}\psi_{1p} &= \frac{C_3}{\alpha_1}\int^{\xi}\sinh\{\alpha_1(\xi - \tilde{t})\}\mathrm{Ai}\{\zeta_1(\tilde{t})E^+\}d\tilde{t} \\ &\quad + \frac{C_4}{\alpha_1}\int^{\xi}\sinh\{\alpha_1(\xi - \tilde{t})\}\mathrm{Ai}\{\zeta_1(\tilde{t})E^-\}d\tilde{t}\end{aligned} \tag{4.3.16}$$

Therefore the general solution of Eq. (4.1.1) is

$$\psi_1(\xi) = C_1\exp(\alpha_1\xi) + C_2\exp(-\alpha_1\xi) + \frac{C_3}{\alpha_1}\int_{\xi^*}^{\xi}\sinh\{\alpha_1(\xi - \tilde{t})\}\mathrm{Ai}\{\zeta_1(\tilde{t})E^+\}d\tilde{t}$$
$$+ \frac{C_4}{\alpha_1}\int_{\xi^*}^{\xi}\sinh\{\alpha_1(\xi - \tilde{t})\}\mathrm{Ai}\{\zeta_1(\tilde{t})E^-\}d\tilde{t} \qquad (4.3.17)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants of integration. ξ^* is chosen for convenience to be such that

$$\zeta_1(\xi^*) = 0 \qquad (4.3.18)$$

Thus ξ^* is equivalent to the turning point of the Airy function for a real variable.

An identical procedure yields the solution of Eq. (4.1.2) as

$$\psi_2(\eta) = C_5\exp(\alpha_2\eta) + C_6\exp(-\alpha_2\eta) + \frac{C_7}{\alpha_2}\int_{\eta^*}^{\eta}\sinh\{\alpha_2(\eta - \tilde{t})\}\mathrm{Ai}\{\zeta_2(\tilde{t})E^+\}d\tilde{t}$$
$$+ \frac{C_8}{\alpha_2}\int_{\eta^*}^{\eta}\sinh\{\alpha_2(\eta - \tilde{t})\}\mathrm{Ai}\{\zeta\;(\tilde{t})E^-\}d\tilde{t} \qquad (4.3.19)$$

where

$$\zeta_2(\eta) = -\frac{\alpha_2^2 + i\alpha_2R_2\{\hat{u}_2(\eta) - c_2\}}{(\alpha_2R_2\dot{\hat{u}}_2)^{2/3}} \qquad (4.3.20)$$

and η^* is such that

$$\zeta_2(\eta^*) = 0 \tag{4.3.21}$$

4.4 Application of Boundary Conditions

In the boundary conditions (4.1.3) - (4.1.12) derivatives of ψ_1 and ψ_2 w.r.t. ξ and η, up to third order, are required. Since ξ and η occur in the limits of integration of Eqs. (4.3.17) and (4.3.19) it is necessary to use Leibniz's rule of differentiation under the integral sign. The results are summarized in Appendix C.

Substituting Eqs. (4.3.17), (4.3.19) and the derivatives in Appendix C into the boundary conditions (4.1.3) - (4.1.11) and performing the required algebraic simplifications, the following equations are obtained:

$$\alpha_1 \exp(-\alpha_1)C_1 + \alpha_1 \exp(\alpha_1)C_2 + I_1C_3 + I_2C_4 = 0 \tag{4.4.1}$$

$$\alpha_1 \exp(-\alpha_1)C_1 - \alpha_1 \exp(\alpha_1)C_2 + I_3C_3 + I_4C_4 = 0 \tag{4.4.2}$$

$$\alpha_2 \exp(\alpha_2)C_5 + \alpha_2 \exp(-\alpha_2)C_6 + I_5C_7 + I_6C_8 = 0 \tag{4.4.3}$$

$$\alpha_2 \exp(\alpha_2)C_5 - \alpha_2 \exp(-\alpha_2)C_6 + I_7C_7 + I_8C_8 = 0 \tag{4.4.4}$$

$$\bar{u}\alpha_1C_1 - \bar{u}\alpha_1C_2 + \bar{u}I_{11}C_3 + \bar{u}I_{12}C_4 - \varepsilon\alpha_2C_5 + \varepsilon\alpha_2C_6 - \varepsilon I_{15}C_7 - \varepsilon I_{16}C_8$$

$$= i\alpha_1(\overline{u\hat{u}_1'} - \varepsilon\dot{\hat{u}}_2) \tag{4.4.5}$$

$$2\overline{u}\alpha_1^2C_1 + 2\overline{u}\alpha_1^2C_2 + \left[\overline{u}Ai\{\zeta_1(0)E^+\} - 2\overline{u}\alpha_1 I_9\right] C_3$$

$$+ \left[\overline{u}Ai\{\zeta_1(0)E^-\} - 2\overline{u}\alpha_1 I_{10}\right] C_4 - 2\overline{\mu}\varepsilon^2\alpha_2^2C_5 - 2\overline{\mu}\varepsilon^2\alpha_2^2C_6$$

$$+ \left[-\overline{\mu}\varepsilon^2 Ai\{\zeta_2(0)E^+\} + 2\overline{\mu}\varepsilon^2\alpha_2 I_{13}\right] C_7 + \left[-\overline{\mu}\varepsilon^2 Ai\{\zeta_2(0)E^-\} + 2\overline{\mu}\varepsilon^2\alpha_2 I_{14}\right] C_8 = 0$$

$$(4.4.6)$$

$$\left[-\frac{1}{\alpha_1}\frac{\overline{u}^2}{\overline{\rho}}\left(\hat{u}_1' - \{\hat{u}_1(0) - c_1\}\alpha_1\right) + \frac{2\overline{u}\alpha_1}{\varepsilon\overline{\mu}R_2}\right] C_1$$

$$+ \left[-\frac{i}{\alpha_1}\frac{\overline{u}^2}{\overline{\rho}}\left(\hat{u}_1' + \{\hat{u}_1(0) - c_1\}\alpha_1\right) - \frac{2\overline{u}\alpha_1}{\varepsilon\overline{\mu}R_2}\right] C_2$$

$$+ \left[-\frac{\overline{u}^2}{\overline{\rho}}\frac{1}{\alpha_1^2R_1} Ai'\{\zeta_1(0)E^+\}\zeta_1'E^+ + \frac{i\overline{u}^2}{\overline{\rho}}\frac{\hat{u}_1'}{\alpha_1^2} I_9 + \left(\frac{i}{\alpha}\frac{\overline{u}^2}{\overline{\rho}}\{\hat{u}_1(0) - c_1\} + \frac{2\overline{u}}{\varepsilon\overline{\mu}R_2}\right) I_{11}\right] C_3$$

$$+ \left[-\frac{\overline{u}^2}{\overline{\rho}}\frac{1}{\alpha_1^2R_1} Ai'\{\zeta_1(0)E^-\}\zeta_1'E^- + \frac{i\overline{u}^2}{\overline{\rho}}\frac{\hat{u}_1'}{\alpha_1^2} I_{10} + \left(\frac{i}{\alpha_1}\frac{\overline{u}^2}{\overline{\rho}}\{\hat{u}_1(0) - c_1\} + \frac{2\overline{u}}{\varepsilon\overline{\mu}R}\right) I_{12}\right] C_4$$

$$+ \left[\frac{i}{\alpha_2}\left(\dot{\hat{u}}_2 - \{\hat{u}_2(0) - c_2\}\alpha_2\right) - \frac{2\alpha_2}{R_2}\right] C_5$$

$$+ \left[\frac{i}{\alpha_2}\left(\dot{\hat{u}}_2 + \{\hat{u}_2(0) - c_2\}\alpha_2\right) + \frac{2\alpha_2}{R_2}\right] C_6$$

$$+ \left[\frac{1}{\alpha_2^2R_2} Ai'\{\zeta_2(0)E^+\}\dot{\zeta}_2E^+ - \frac{i\dot{\hat{u}}_2}{\alpha_2^2} I_{13} - \left(\frac{i}{\alpha_2}\{\hat{u}_2(0) - c_2\} + \frac{2}{R_2}\right) I_{15}\right] C_7$$

$$+ \left[\frac{1}{\alpha_2^2R_2} Ai'\{\zeta_2(0)E^-\}\dot{\zeta}_2E^- - \frac{i\dot{\hat{u}}_2}{\alpha_2^2} I_{14} - \left(\frac{i}{\alpha_2}\{\hat{u}_2(0) - c_2\} + \frac{2}{R_2}\right) I_{16}\right] C_8$$

$$= -\frac{\overline{u}^2}{\overline{\rho}}\left(\alpha^2W^2 + \frac{1}{F^2}\right) \qquad (4.4.7)$$

$$\alpha_1 C_1 + \alpha_1 C_2 - I_9 C_3 - I_{10} C_4 = i\alpha_1^2\{\hat{u}_1(0) - c_1\} \qquad (4.4.8)$$

$$\alpha_2 C_5 + \alpha_2 C_6 - I_{13} C_7 - I_{14} C_8 = i\alpha_1\alpha_2\{\hat{u}_2(0) - c_2\} \qquad (4.4.9)$$

where the integrals I_1 through I_{16} are defined in Appendix E.

Eqs. (4.4.1) - (4.4.9) form a system of eight linear algebraic equations of the type

$$\left[A(c_1)\right]\{C\} = \{V(c_1)\} \qquad (4.4.10)$$

where $\left[A(c_1)\right]$ is the coefficient matrix of the left hand sides and $\{V(c_1)\}$ is the column vector of the right hand sides. The remaining equation (4.4.9) can be written in the form,

$$G\left[c_1;\{C(c_1)\}\right] = \alpha_2 C_5 + \alpha_2 C_6 - I_{13} C_7 - I_{14} C_8 - i\alpha_1\alpha_2(\hat{u}_2(0) - c_2) = 0 \qquad (4.4.11)$$

or more compactly

$$G\left[c_1;\{C(c_1)\}\right] = \{C(c_1)\}^T\{U(c_1)\} - i\alpha_1\alpha_2(\hat{u}_2(0) - c_2) = 0 \qquad (4.4.12)$$

where

$$\{C(c_1)\} = \left[A(c_1)\right]^{-1}\{V(c_1)\} \qquad (4.4.13)$$

and

$$\{U(c_1)\}^T = \{0 \quad 0 \quad 0 \quad 0 \quad \alpha_2 \quad \alpha_2 \quad -I_{13} \quad -I_{14}\} \qquad (4.4.14)$$

In Eqs. (4.4.11) and (4.4.12) c_2 is given by Eq. (4.1.15), G is the characteristic function of Sec. 2.7 and c_1 is the eigenvalue. The problem of determining stability of the interface is thus reduced to finding the points in the complex c_1 plane at which G vanishes. Eq. (4.4.12) can be expressed in the functional form

$$G\ (\alpha_1, \varepsilon, \overline{\mu}_1 \overline{\rho}, R_2, W_1 F; c_1) = 0 \qquad (4.4.16)$$

For the present physical problem G is an analytic function of the above parameters and the mathematical problem reduces to finding the zeros of an analytic function.

4.5 Outline of the Eigenvalue Iteration Procedure

The zeros of the characteristic function G must be determined numerically and the major steps in this procedure are listed below.

(i) Guess c_1. A method for obtaining a good guess value is described in the next section.

(ii) For given values of α_1, ε, $\overline{\mu}$, $\overline{\rho}$, R_2, W, F and the guess value c_1, evaluate the integrals $I_1 - I_{16}$ using a suitable numerical procedure. This is described in Appendix G.

(iii) Generate the coefficient matrix [A] and obtain its inverse. The inverse was obtained using the routine MINV in the IBM Scientific Subroutine Package after making minor changes to handle complex numbers. The constants of integration were conveniently scaled to avoid overflows in MINV. In order to check the accuracy of matrix inversion the eigenvalue was obtained using MINV alone and by applying a first order correction to $[A]^{-1}$. The difference between the eigenvalues obtained by these two methods was negligibly small.

Also compute the right hand side column vector $\{V\}$.

(iv) Determine the constants of integration i.e. the vector $\{C\}$ by carrying out the matrix multiplication $[A]^{-1}\{V\}$.

(v) Compute G in Eq. (4.4.11), it should be close to zero if this equation is satisfied.

(vi) If Eq. (4.4.11) is not satisfied calculate an improved value of c_1 by using a suitable technique. In the present work, the Newton-Raphson method was used for this purpose. This iterative method requires computation of the first derivative of G. w.r.t. c_1. Details of the Newton-Raphson method are given in Appendix I.

(vii) Compare successive values of c_1 for convergence within a prescribed tolerance on real and imaginary parts. Repeat steps (ii) - (vi) until desired convergence is reached.

4.6 Generation of an Initial Guess for c_1

Since the Newton-Raphson method is sensitive to the initial guess it is necessary to know an approximate value of c_1. This value could be determined either on a mathematical basis or a physical basis. The small perturbation method of Sec. 4.1 can be used as a mathematical basis. However, it leads to only one zero of the characteristic function G. It was found in the present investigation that a number of zeros of G are possible (see Chapter 6 for details). Hence a more reliable approach to obtain the guess is necessary. As far as the physical basis is concerned the works of Lock[27] and Landahl[10] suggest that the real part of c_1 or the phase speed should be close to the speed of propagation of free surface waves. Guessing only the real part accurately, however, is not sufficient because the imaginary part is of significance as well. It would be adequate to guess only the real part if the investigation were to be confined to neutral stability analysis. In the latter case the imaginary part of c_1 is necessarily zero.

The above discussion shows that a simple clear-cut solution is apparently not possible. An attempt was made to determine at least the number of zeros of G inside a closed contour in the complex c_1 plane. This was done using the 'argument principle' which states that for a regular analytic function G(Z) within a closed contour C the number of zeros of G within C is given by

$$\frac{1}{2\pi i} \oint_C \frac{G'(Z)}{G(Z)}\, dZ \tag{4.6.1}$$

where it is assumed that G does not vanish on C.

Numerical evaluation of the above integral as the limit of a sum proved to be a very difficult task. This was due to the fact that the integrand in (4.6.1) is highly oscillatory and undergoes large changes of magnitude. Consequently, either graphical or purely numerical determination of the number of zeros in this manner would be extremely difficult unless a very fine step size along the contour is employed. The latter choice, of course, leads to large computation times. Therefore, this course also had to be abandoned.

The only choice available is to determine the zeros of Re(G) and Im(G) separately and then to obtain their common zeros. This can be done graphically as follows. First a suitable interval on c_1 is chosen. Recalling that the phase speed ω/k was made dimensionless w.r.t. the interface velocity {Eq.(2.8.14)}, it is seen that $c_{1r} = 1$ corresponds to a critical point at the interface. Thus disturbances which propagate with phase speed greater than the interface velocity lie in the interval $1 \leq c_{1r} \leq 1/\bar{u}$. Now Eq. (4.1.17) shows that for thin liquid layers and typically small viscosity ratios $\epsilon\bar{\mu} << 1$ and thus $1/\bar{u} \simeq \epsilon\bar{\mu}$ is large. Hence the gas side interval on c_{1r} is very large compared to the liquid side $(0 \leq c_{1r} \leq 1)$. The critical points of interest, however, are those that lie near the interface. In other words, relatively slow moving disturbances near the interface are of interest. The reason being that in a real physical situation slow moving disturbances are more likely to be triggered. In conclusion, it is sufficient to consider a typical interval $0 \leq c_{1r} \leq 4$.

The next step is to choose a suitable interval on c_{1i}. It is not possible in this case to offer an argument similar to the previous one. Therefore, $0 \leq |c_{1i}| \leq 1$ is chosen as a start. G is then calculated at a number of points inside the unit rectangle (e.g., at intervals of 0.1 along c_{1r} and c_{1i}). Then Re(G) is plotted against Re(c_1) with Im(c_1) as a parameter and the points of intersection on the x-axis are determined. Similar plots are made for Im(G) and its zeros are also determined. It is usually observed that as one proceeds from $c_{1i} = 0$ to $|c_{1i}| = 1$ the trend of the curves shows that beyond a certain c_{1i} there are no intersections on the x-axis. This fact determines the upper limit on c_{1i}. Finally, common zeros of Re(G) and Im(G) can be roughly determined. These approximate values of c_1 serve as initial guesses for Newton-Raphson iteration.

An illustrative example of the above procedure is included in Chapter VI.

CHAPTER V

SOLUTION OF THE MASS TRANSFER PROBLEM

5.1 Linear Approximation of Exponential Steady-State Profiles

The present investigation concerns itself with small values of mass transfer rates. Hence it is assumed that

$$R_h << 1 \quad \text{and} \quad R_\delta << 1 \tag{5.1.1}$$

Since $|\xi|$ and $|\eta|$ are always less than unity the exponentials in Eqs. (3.7.3) and (3.7.5) can be expanded in a Taylor series to give (up to first order)

$$\hat{u}_1(\xi) = 1 + \xi \, , \; -1 \leq \xi \leq 0 \tag{5.1.2}$$

and

$$\hat{u}_2(\eta) = \frac{R_h + R_\delta \eta}{R_h + R_\delta} \, , \; 0 \leq \eta \leq 1 \tag{5.1.3}$$

It may be verified that the linear profiles in the last two equations reduce to Eqs. (2.8.3) and (2.8.4) with the help of Eqs. (3.7.8) and (3.7.9). The mass fraction profile of Eq. (3.7.7) reduces to

$$\hat{\chi}(\eta) = R_\delta(1 - \eta), \quad 0 \leq \eta \leq 1 \tag{5.1.4}$$

In order to linearize (3.7.4) it needs to be assumed in addition to (5.1.1) that

$$R_h Pr_1 << 1 \tag{5.1.5}$$

Typically, $Pr_1 \simeq 0(10)$ and this requires R_h to be smaller than 0.01.

The linearized forms of Eqs. (3.7.4) and (3.7.6) are

$$\hat{T}_1 = 1 + \frac{R_h Pr_1 \left[\bar{c}_p \left(1 - \frac{T_w}{T_e} \right) - \frac{\ell}{c_{p1} T_e} R_\delta \right]}{R_h Pr_1 \left[\bar{c}_p - \frac{\ell}{c_{p1} T_e} R_\delta \right] + R_\delta \frac{T_w}{T_e}} \xi$$

$$-1 \leq \xi \leq 0 \tag{5.1.6}$$

$$\hat{T}_2 = \frac{\bar{c}_p R_h Pr_1 + R_\delta \left[\frac{T_w}{T_e} - \frac{\ell}{c_{p1} T_e} R_h Pr_1 \right] - R_\delta \left[\frac{T_w}{T_e} - \frac{\ell}{c_{p1} T_e} R_h Pr_1 - 1 \right] \eta}{R_\delta + \bar{c}_p R_h Pr_1}$$

$$0 \leq \eta \leq 1 \tag{5.1.7}$$

Finally, linearized expressions for the interface quantities in Eqs. (3.7.13), (3.7.14) and (3.7.7) are

$$\bar{u} = \frac{R_h}{R_\delta + R_h} \tag{5.1.8}$$

$$\bar{T} = \frac{R_\delta \frac{T_w}{T_e} + R_h Pr_1 \left[\bar{c}_p - \frac{\ell}{c_{p1} T_e} R_\delta \right]}{R_\delta + \bar{c}_p R_h Pr_1} \tag{5.1.9}$$

$$\bar{\chi} = R_\delta \tag{5.1.10}$$

5.2 Mathematical Statement of the Eigenvalue Problem

The mathematical statement of the mass transfer problem of Sec. 3.7 is given below for the case of the linear steady-state profiles described in Sec. 5.1.

Governing equations:

1. $$\psi_1^{iv} - R_h\psi_1''' - 2\alpha_1^2\psi_1'' + \alpha_1^2 R_h\psi_1' + \alpha_1^4\psi_1$$

$$= i\alpha_1 R_1(\psi_1'' - \alpha^2\psi_1)(\hat{u}_1(\xi) - c_1) \qquad (5.2.1)$$

2. $$\theta_1'' - Pr_1 R_h\theta_1' - \{\alpha_1^2 + i\alpha_1 Pr_1 R_1(\hat{u}_1(\xi) - c_1\}\theta_1 = Pr_1 R_1\hat{T}_1'\psi_1 \qquad (5.2.2)$$

3. $$\ddddot{\psi}_2 - R_\delta\dddot{\psi}_2 - 2\alpha_2^2\ddot{\psi}_2 + \alpha_2^2 R_\delta\dot{\psi}_2 + \alpha_2^4\psi_2$$

$$= i\alpha_2 R_2(\ddot{\psi}_2 - \alpha_2^2\psi_2)(\hat{u}_2(\eta) - c_2) \qquad (5.2.3)$$

4. $$\ddot{\theta}_2 - R_\delta\dot{\theta}_2 - \{\alpha_2^2 + i\alpha_2 R_2(\hat{u}_2(\eta) - c_2)\}\theta_2 = R_2\dot{\hat{T}}_2\psi \qquad (5.2.4)$$

5. $$\ddot{\chi} - R_\delta\dot{\chi} - \{\alpha^2 + i\alpha_2^2 R_2(\hat{u}_2(\eta) - c_2)\}\chi = R_2\dot{\hat{\chi}}\psi_2 \qquad (5.2.5)$$

Boundary conditions:

1. $$\psi_1(-1) = 0 \qquad (5.2.6)$$

2. $$\psi_1'(-1) = 0 \qquad (5.2.7)$$

3. $$\psi_2(1) = 0 \qquad (5.2.8)$$

4. $$\dot{\psi}_2(1) = 0 \qquad (5.2.9)$$

5. $$\overline{u}\left[\psi_1'(0) - i\alpha_1\hat{u}_1' + \alpha_1^2\frac{R_h}{R_1}\right] = \varepsilon\left[\dot{\psi}_2(0) - i\alpha_2\dot{\hat{u}}_2 + \alpha_1\alpha_2\frac{R_\delta}{R_2}\right] \qquad (5.2.10)$$

6. $$\overline{u}\left[\psi_1''(0) + \alpha_1^2\psi_1(0)\right] = \overline{\mu}\varepsilon^2\left[\ddot{\psi}_2(0) + \alpha_2^2\psi_2(0)\right] \quad (5.2.11)$$

$$\frac{1}{\alpha_2^2 R_2}\left[\dddot{\psi}_2(0) - \alpha_2^2\dot{\psi}_2(0)\right] - \frac{\overline{u}^2}{\overline{\rho}}\frac{1}{\alpha_1^2 R_1}\left[\psi_1'''(0) - \alpha_1^2\psi_1'(0)\right]$$

7. $$+\frac{\overline{u}^2}{\overline{\rho}}\frac{1}{\alpha_1^2}\frac{R_h}{R_1}\psi_1''(0) - \frac{1}{\alpha_2^2}\frac{R_\delta}{R_2}\ddot{\psi}_2(0) + i\left[\frac{1}{\alpha_2}\{\psi_2(0)\dot{\hat{u}}_2 - (\hat{u}_2(0) - c_2)\dot{\psi}_2(0)\}\right.$$

$$\left.-\frac{\overline{u}^2}{\overline{\rho}}\frac{1}{\alpha_1}\{\psi_1(0)\hat{u}_1' - (\hat{u}_1(0) - c_1)\psi_1'(0)\}\right] - \frac{2}{\varepsilon\overline{\mu}R_2}\{\varepsilon\overline{\mu}\dot{\psi}_2(0) - \overline{u}\psi_1'(0)\}$$

$$+ 2\left[\frac{R_\delta}{R_2}\psi_2(0) - \frac{\overline{u}^2}{\overline{\rho}}\frac{R_h}{R_1}\psi_1(0)\right] = -\frac{\overline{u}^2}{\overline{\rho}}\left(\alpha_1^2 W^2 + \frac{1}{F^2}\right) \quad (5.2.12)$$

8. $$\overline{\rho}\left[\psi_2(0) - i\alpha_1(\hat{u}_2(0) - c_2)\right] = \overline{u}\left[\psi_1(0) - i\alpha_1(\hat{u}_1(0) - c_1)\right] \quad (5.2.13)$$

9. $$R_2\{1 - \hat{\chi}(0)\}\left[\psi_2(0) - i\alpha_1(\hat{u}_2(0) - c_2)\right] - R_\delta\chi(0) + \dot{\chi}(0) = 0 \quad (5.2.14)$$

10. $$\theta_1(0) = 0 \quad (5.2.15)$$

11. $$\theta_2(1) = 0 \quad (5.2.16)$$

12. $$\theta_2(0) - \overline{T}\theta_1(0) = \overline{T}\hat{T}_1' - \varepsilon\dot{\hat{T}}_2 \quad (5.2.17)$$

13. $$\dot{\theta}_2(0) - \frac{\overline{T}}{\varepsilon\overline{\lambda}}\theta_1'(0) = \frac{R_2\overline{T}\Lambda}{\overline{c}_p}\left[\psi_2(0) - i\alpha_1(\hat{u}_2(0) - c_2)\right] \quad (5.2.18)$$

14. $$\chi(1) = 0 \quad (5.2.19)$$

15. $$\frac{1}{E}\left[\frac{1}{\alpha_2^2 R_2}\{\dddot{\psi}_2(0) - R_\delta\ddot{\psi}_2(0) - \alpha_2^2\dot{\psi}_2(0)\} - \frac{1}{i\alpha_2}\left(\dot{\hat{u}}_2\psi_2(0) - \dot{\psi}_2(0)\{\hat{u}_2(0) - c_2\}\right)\right]$$

$$+\frac{\chi(0)}{\hat{\chi}(0)} = \overline{\overline{R}}\overline{T}\frac{\theta_2(0)}{\hat{T}_2^2(0)} \quad (5.2.20)$$

where $\hat{u}_1(\xi)$, $\hat{u}_2(\eta)$, $\hat{\chi}(\eta)$, $\hat{T}_1(\eta)$ and $\hat{T}_2(\eta)$ are given by Eqs. (5.1.2), (5.1.3), (5.1.4), (5.1.6) and (5.1.7) respectively. $\bar{u}$ and $\bar{T}$ are the expressions (5.1.8) and (5.1.9) respectively. Also, the relations (4.1.14) - (4.1.17) hold for the mass transfer problem as well. The evaluation of mass transfer Reynolds numbers R_h and R_δ is the subject of Sec. 5.3.

5.3 Evaluation of Mass Transfer Reynolds Numbers

It was mentioned in Sec. 3.3 that the mass transfer Reynolds numbers R_δ and R_h have to be determined iteratively. This procedure is discussed briefly in this section. Referring to Eqs. (3.3.6) and letting

$$x = \exp(R_\delta Pr_2) = \exp(R_\delta) \text{ for } Pr_2 = 1 \tag{5.3.1}$$

the result is

$$0 = H(x) = \ln \frac{p_e}{p_{ref}} - \frac{\ell}{RT_{ref}} + \ln\left(1 - \frac{1}{x}\right) + \frac{\ell}{RT_e} \cdot \frac{(x-1) + \bar{c}_p\left(1 - \frac{1}{x^n}\right)}{\bar{c}_p\left(1 - \frac{1}{x^n}\right) + (x-1)\left[\frac{T_w}{T_e} - \frac{\ell}{c_{p1}T_e}\left(1 - \frac{1}{x^n}\right)\right]} \tag{5.3.2}$$

$$n = \frac{h}{\delta}\frac{\mu_2}{\mu_1}\frac{Pr_1}{Pr_2} = \varepsilon\bar{\mu}Pr_1 \tag{5.3.3}$$

In most cases of interest $\varepsilon << 1$ and $\overline{\mu} << 1$ and Pr_1 is typically less than 10 so that $n << 1$. In order to evaluate R_δ it is necessary to solve the transcendental equation (5.3.2). This equation was solved using the Newton-Raphson method which requires specification of a guess for x. It may be verified that the function H(x) has only one zero when $n << 1$ and $\ell/c_{p1}T_e > T_w/T_e$. The latter condition holds in most practical problems of interest. Fig. 4 illustrates the behavior of the function H(x) as a function of x and it is seen that $x = 1$ is the obvious initial guess. Once x is determined R_δ is known from (5.3.1) and R_h is determined from the relation $R_h = \varepsilon\overline{\mu}R_\delta$.

5.4 General Solutions of the Modified Orr-Sommerfeld Equations

Eq. (5.2.1) can be written in the form

$$(\psi_1'' - \alpha_1^2\psi_1)'' - \alpha_1^2(\psi_1'' - \alpha_1^2\psi_1)' - R_h(\psi_1'' - \alpha_1^2\psi_1) - i\alpha_1R_1(\psi_1'' - \alpha_1^2\psi_1)(\hat{u}_1(\xi) - c_1) = 0 \quad (5.4.1)$$

Let

$$\psi_1'' - \alpha_1^2\psi_1 = w_1(\xi) \quad (5.4.2)$$

Then Eq. (5.4.1) becomes

$$w_1'' - R_hw_1' - \left\{\alpha_1^2 + i\alpha_1R_1(\hat{u}_1(\xi) - c_1)\right\}w_1 = 0 \quad (5.4.3)$$

In order to reduce the above equation to an Airy differential equation it is necessary to eliminate the first derivative term. Thus let

$$w_1(\xi) = S_1(\xi)H_1(\xi) \tag{5.4.4}$$

Substitution of Eq. (5.4.4) into (5.4.3) leads to

$$H_1'' + \frac{(2S_1' - R_h S_1)}{S_1} H_1' + \frac{\left[S_1'' - R_h S_1' - \{\alpha_1^2 + i\alpha_1 R_1(\hat{u}_1(\xi) - c_1)\}\right]}{S_1} H_1 = 0 \tag{5.4.5}$$

Eq. (5.4.5) indicates that S_1 should be chosen such that

$$2S_1' - R_h S_1 = 0$$

or

$$S_1 = e^{R_h \xi/2} \tag{5.4.6}$$

With this choice of S_1 Eq. (5.4.5) takes the form

$$H_1'' - \left[\frac{R_h^2}{4} + \alpha_1^2 + i\alpha_1 R_1(\hat{u}_1(\xi) - c_1)\right] H_1 = 0 \tag{5.4.7}$$

Defining

$$\zeta_1(\xi) = -\frac{\frac{R_h^2}{4} + \alpha_1^2 + i\alpha_1 R_1(\hat{u}_1(\xi) - c_1)}{(\alpha_1 R_1 \hat{u}_1')^{2/3}} \tag{5.4.8}$$

where $\hat{u}_1'$ is a constant and carrying out the transformation of independent variable in Eq. (5.4.7), the result is

$$\frac{d^2h_1}{d\zeta_1^2} - \zeta_1 h_1 = 0 \tag{5.4.9}$$

where

$$h_1(\zeta_1) = H_1\{\xi(\zeta_1)\} \tag{5.4.10}$$

Eq. (5.4.9) is the Airy differential equation encountered earlier in Sec. 4.3. The experience gained in the solution of the zero mass transfer problem suggests the following solution of (5.4.9),

$$h_1(\zeta_1) = C_3 Ai(\zeta_1 E^+) + C_4 Ai(\zeta_1 E^-)$$

or

$$H_1(\xi) = C_3 Ai(\zeta_1(\xi)E^+) + C_4 Ai(\zeta_1(\xi)E^-) \tag{5.4.11}$$

Where $E^{\pm}$ is given by Eq. (4.3.12)

From Eq. (5.4.4)

$$w_1(\xi) = e^{R_h \xi/2}\left[C_3 Ai\{\zeta_1(\xi)E^+\} + C_4 Ai\{\zeta_1(\xi)E^-\}\right]$$

and hence Eq. (5.4.2) assumes the form

$$\psi_1'' - \alpha_1^2\psi_1 = e^{R_h \xi/2}\left[C_3 Ai\{\zeta_1(\xi)E^+\} + C_4 Ai\{\zeta_1(\xi)E^-\}\right] \tag{5.4.12}$$

The homogeneous solution of the above equation is

$$\psi_{1H} = C_1 e^{\alpha_1 \xi} + C_2 e^{-\alpha_1 \xi} \tag{5.4.13}$$

and the particular solution obtained by the method of variation of parameters is

$$\psi_{1p} = \frac{C_3}{\alpha_1}\int^{\xi} e^{R_h \tilde{t}/2} \sinh\{\alpha_1(\xi - \tilde{t})\} Ai\{\zeta_1(\tilde{t})E^+\}d\tilde{t}$$

$$+ \frac{C_4}{\alpha_1}\int^{\xi} e^{R_h \tilde{t}/2} \sinh\{\alpha_1(\xi - \tilde{t})\} Ai\{\zeta_1(\tilde{t})E^-\}d\tilde{t} \tag{5.4.14}$$

Finally, the general solution of Eq. (5.2.1) is

$$\psi_1(\xi) = C_1 e^{\alpha_1 \xi} + C_2 e^{-\alpha_1 \xi} + \frac{C_3}{\alpha_1} \int_{\xi^*}^{\xi} e^{R_h \tilde{t}/2} \sinh\{\alpha_1(\xi - \tilde{t})\} \mathrm{Ai}\{\zeta_1(\tilde{t}) E^{+}\} d\tilde{t}$$

$$+ \frac{C_4}{\alpha_1} \int_{\xi^*}^{\xi} e^{R_h \tilde{t}/2} \sinh\{\alpha_1(\xi - \tilde{t})\} \mathrm{Ai}\{\zeta_1(\tilde{t}) E^{-}\} d\tilde{t} \qquad (5.4.15)$$

where C_1, C_2, C_3, and C_4 are arbitrary constants of integration and ξ^* is selected such that

$$\zeta_1(\xi^*) = 0 \qquad (5.4.16)$$

It should be noted that the same notation ζ_1 is used in both zero mass transfer and mass transfer cases, however, ζ_1 has different definitions as seen from Eqs. (4.3.4) and (5.4.8).

To obtain the general solution of (5. 2.3) the procedure described above is repeated to yield the result

$$\psi_2(\eta) = C_5 e^{\alpha_2 \eta} + C_6 e^{-\alpha_2 \eta} + \frac{C_7}{\alpha_2} \int_{\eta^*}^{\eta} e^{R_\delta \tilde{t}/2} \sinh\{\alpha_2(\eta - \tilde{t})\} \mathrm{Ai}\{\zeta_2(\tilde{t}) E^{+}\} d\tilde{t}$$

$$+ \frac{C_8}{\alpha_1} \int_{\eta^*}^{\eta} e^{R_\delta \tilde{t}/2} \sinh\{\alpha_2(\eta - \tilde{t})\} \mathrm{Ai}\{\zeta_2(\tilde{t}) E^{-}\} d\tilde{t} \qquad (5.4.17)$$

where

$$\zeta_2(\tilde{t}) = - \frac{\frac{R_\delta^2}{4} + \alpha_2^2 + i\alpha_2 R_2 (\hat{u}_2(\tilde{t}) - c_2)}{(\alpha_2 R_2 \dot{\hat{u}}_2)^{2/3}} \qquad (5.4.18)$$

5.5 General Solutions of Temperature and Concentration Perturbation Equations

(i) Solution of temperature perturbation equation for liquid:

Consider the homogeneous part of Eq. (5.2.2), viz.,

$$\theta_1'' - Pr_1 R_h \theta_1' - \{\alpha_1^2 + i\alpha_1 Pr_1 R_1 (\hat{u}_1(\xi) - c_1)\}\theta_1 = 0 \qquad (5.5.1)$$

Following the procedure of Sec. 5.4 the first derivative term is eliminated from (5.5.1) and the resulting equation is converted into an Airy differential equation. The latter is solved as before to give the homogeneous solution

$$\theta_{1H} = e^{R_h Pr_1 \xi/2}\,\{C_9 Ai(z_1(\xi)E^+) + C_{10} Ai(z_1(\xi)E^-)\} \qquad (5.5.2)$$

where

$$z_1(\xi) = -\frac{\dfrac{R_h^2 Pr_1^2}{4} + \alpha_1^2 + i\alpha_1 Pr_1 R_1(\hat{u}_1(\xi) - c_1)}{(\alpha_1 Pr_1 R_1 \hat{u}_1')^{2/3}} \qquad (5.5.3)$$

Consider now the particular solution of Eq. (5.2.2). Substitution for ψ_1 from Eq. (5.4.15) into (5.2.2) yields

$$\theta_1'' - Pr_1 R_h \theta_1' - \{\alpha^2 + i\alpha_1 Pr_1 R_1 (\hat{u}_1(\xi) - c_1)\theta_1$$

$$= Pr_1 R_1 \hat{T}_1' \left\{ C_1 e^{\alpha_1 \xi} + C_2 e^{-\alpha_1 \xi} + \frac{C_3}{\alpha_1}\int_{\xi^*}^{\xi} e^{R_h \tilde{t}/2} \sinh\{\alpha_1(\xi - \tilde{t})\} Ai\{\zeta_1(\tilde{t})E^+\} d\tilde{t} \right.$$

$$\left. + \frac{C_4}{\alpha_1}\int_{\xi^*}^{\xi} e^{R_h \tilde{t}/2} \sinh\{\alpha_1(\xi - \tilde{t})\} Ai\{\zeta_1(\tilde{t})E^-\} d\tilde{t} \right\} \qquad (5.5.4)$$

where $\hat{T}_1'$ is constant for the linear temperature profile in Eq. (5.1.6). The general solution of Eq. (5.5.4) may be written

$$\theta_1 = \theta_{1H} + Pr_1 R_1 \hat{T}_1' \{C_1 \bar{J}_1 + C_2 \bar{J}_2 + C_3 \bar{J}_3 + C_4 \bar{J}_4\} \tag{5.5.5}$$

where $\bar{J}_1$, $\bar{J}_2$, $\bar{J}_3$, and $\bar{J}_4$ are particular integrals obtained by the method of variation of parameters,

i.e.

$$\bar{J}_{1,2} = \int_{\xi^*}^{\xi} \frac{y_1(\tilde{t})y_2(\xi) - y_2(\tilde{t})y_1(\xi)}{W(\tilde{t})} \; e^{\pm\alpha_1 \tilde{t}} d\tilde{t} \tag{5.5.6}$$

$$\bar{J}_{3,4} = \int_{\xi^*}^{\xi} \frac{y_1(\tilde{t})y_2(\xi) - y_2(\tilde{t})y_1(\xi)}{W(\tilde{t})} \int_{\tilde{t}^*}^{\tilde{t}} e^{R_h \tilde{\tau}/2} \; \sinh\{\alpha_1(\tilde{t} - \tilde{\tau})\} Ai\{\zeta_1(\tilde{\tau})E^{\pm}\} d\tilde{\tau} d\tilde{t} \tag{5.5.7}$$

where

$$y_{1,2} = e^{R_h Pr_1 \xi/2} Ai\{z_1(\xi)E^{\pm}\} \tag{5.5.8}$$

and the Wronskian of $y_{1,2}$ is defined by

$$W\{y_1(\tilde{t}), y_2(\tilde{t})\} = y_1(\tilde{t}) \frac{dy_2(\tilde{t})}{d\tilde{t}} - y_2(\tilde{t}) \frac{dy_1(\tilde{t})}{d\tilde{t}} \tag{5.5.9}$$

Combining Eqs. (5.5.8) and (5.5.9) and simplifying, it may be verified that

$$W\{y_1(\tilde{t}), y_2(\tilde{t})\} = e^{R_h Pr_1 \tilde{t}} \frac{dz_1(\tilde{t})}{d\tilde{t}} W\{Ai[z_1(\tilde{t})E^+], Ai[z_1(\tilde{t})E^-]\} \tag{5.5.10}$$

Differentiating $z_1(\tilde{t})$ in Eq. (5.5.3) w.r.t. $\tilde{t}$

$$\frac{dz_1(\tilde{t})}{d\tilde{t}} = - i(\alpha_1 Pr_1 R_1 \hat{u}_1')^{2/3} \tag{5.5.11}$$

Also, the Wronskian in (5.5.10) is given by (Ref. 44)

$$W[Ai\{z_1(\tilde{t})E^+\}, Ai\{z_1(\tilde{t})E^-\}] = \frac{i}{2\pi} \tag{5.5.12}$$

Introducing the expressions in Eqs. (5.5.11) and (5.5.12) into Eq. (5.5.10) one gets

$$W(\tilde{t}) = \frac{1}{2\pi}(\alpha_1 Pr_1 R_1 \hat{u}_1')^{2/3} e^{R_h Pr_1 \tilde{t}} \tag{5.5.13}$$

It is now necessary to evaluate the integrals (5.5.6) and (5.5.7) with the help of Eqs. (5.5.8) and (5.5.13). The end results of these manipulations are given below.

$$\bar{J}_{1,2} = \frac{2\pi e^{R_h Pr_1 \xi/2}}{(\alpha_1 Pr_1 R_1 \hat{u}_1')^{1/3}} \int_{\xi^*}^{\xi} \Big\{ Ai\{z_1(\tilde{t})E^+\}Ai\{z_1(\xi)E^-\}$$

$$-Ai\{z_1(\tilde{t})E^-\}Ai\{z_1(\xi)E^+\} \Big\} e^{(\pm\alpha_1 - \frac{R_h Pr_1}{2})\tilde{t}} \, d\tilde{t} \tag{5.5.14}$$

and

$$\bar{J}_{3,4} = \frac{2\pi e^{R_h Pr_1 \xi/2}}{(\alpha_1 Pr_1 R_1 \hat{u}_1')^{1/3}} \int_{\xi^*}^{\xi} e^{-R_h Pr_1 \tilde{t}/2} \Big[Ai\{z_1(\tilde{t})E^+\} Ai\{z_1(\tilde{\xi})E^-\}$$

$$- Ai\{z_1(\tilde{t})E^-\} Ai\{z_1(\xi)E^+\} \Big] X$$

$$\int_{\tilde{t}^*}^{\tilde{t}} e^{R_h \tilde{\tau}/2} \sinh\{\alpha_1(\tilde{t} - \tilde{\tau})\} Ai\{\zeta_1(\tilde{\tau})E^{\pm}\} d\tilde{\tau} d\tilde{t} \qquad (5.5.15)$$

Finally, the general solution in Eq. (5.5.5) can be written in a slightly different form as

$$\theta_1(\xi) = e^{R_h Pr_1 \xi/2} \Bigg[C_9 Ai\{z_1(\xi)E^+\} + C_{10} Ai\{z_1(\xi)E^-\}$$

$$+ \frac{2\pi Pr_1 R_1 \hat{T}_1'}{(\alpha_1 Pr_1 R_1 \hat{u}_1')^{1/3}} \Big\{ C_1 J_1(\xi) + C_2 J_2(\xi) + C_3 J_3(\xi) + C_4 J_4(\xi) \Big\} \Bigg] \qquad (5.5.16)$$

where

$$J_{1,2}(\xi) = \int_{\xi^*}^{\xi} e^{(\pm\alpha_1 - \frac{R_h Pr_1}{2})\tilde{t}} F_1(\tilde{t};\xi) d\tilde{t} \qquad (5.5.17)$$

$$J_{3,4}(\xi) = \int_{\xi^*}^{\xi} e^{-R_h Pr_1 \tilde{t}/2} F_1(\tilde{t};\xi) \int_{\tilde{t}^*}^{\tilde{t}} e^{R_h \tilde{\tau}/2} \sinh\{\alpha_1(\tilde{t} - \tilde{\tau})\} Ai\{\zeta_1(\tilde{\tau})E^{\pm}\} d\tilde{\tau} d\tilde{t} \qquad (5.5.18)$$

with

$$F_1(\tilde{t};\xi) = Ai\{z_1(\tilde{t})E^+\} Ai\{z_1(\xi)E^-\} - Ai\{z_1(\xi)E^+\} Ai\{z_1(\tilde{t})E^-\} \qquad (5.5.19)$$

and

$$\zeta_1(t^*) = 0 \tag{5.5.20}$$

$$z_1(\xi^*) = 0 \tag{5.5.21}$$

(ii) Solution of temperature perturbation equation in gas-vapor:

Following the procedure described above, the general solution of Eq. (5.2.4) is

$$\theta_2(\eta) = e^{R_\delta \eta/2}\left[C_{11}Ai\{z_2(\eta)E^+\} + C_{12}Ai\{z_2(\eta)E^-\} + \frac{2\pi R_2\dot{\hat{T}}_2}{(\alpha_2 R_2 \dot{\hat{u}}_2)^{1/3}}\left\{C_5J_5(\eta) + C_6J_6(\eta) + C_7J_7(\eta) + C_8J_8(\eta)\right\}\right] \tag{5.5.22}$$

where

$$J_{5,6} = \int_{\eta^*}^{\eta} e^{(\pm\alpha_2 - \frac{R_\delta}{2})\tilde{t}}\; F_2(\tilde{t};\eta)d\tilde{t} \tag{5.5.23}$$

$$J_{7,8} = \int_{\eta^*}^{\eta} e^{-R_\delta \tilde{t}/2}\; F_2(\tilde{t};\eta)\int_{\tilde{t}^*}^{\tilde{t}} e^{R_\delta\tilde{\tau}/2}\; \sinh\{\alpha_2(\tilde{t} - \tilde{\tau})\}Ai\{\zeta_2(\tilde{\tau})E^{\pm}\}d\tilde{\tau}d\tilde{t} \tag{5.5.24}$$

with

$$F_2(\tilde{t};\eta) = Ai\{z_2(\tilde{t})E^+\}Ai\{z_2(\eta)E^-\} - Ai\{z_2(\eta)E^+\}Ai\{z_2(\tilde{t})E^-\} \tag{5.5.25}$$

and

$$z_2(t^*) = \zeta_2(\eta^*) = 0 \qquad (5.5.26)$$

It should be noted that $z_2(t)$ and $\zeta_2(t)$ are identical since the :andtl number for the gas-vapor is assumed unity (See Eq. (5.4.18)).

(iii) Solution of the concentration perturbation equation

Applying the procedure in (i) to Eq. (5.2.5) would give the general solution

$$\chi(\eta) = e^{R_\delta \eta/2}\Bigg[C_{13}\mathrm{Ai}\{z_2(\eta)E^+\} + C_{14}\mathrm{Ai}\{z_2(\eta)E^-\} + \frac{2\pi R_2\hat{\dot{\chi}}}{(\alpha_2 R_2 \hat{\dot{u}}_2)^{1/3}}\Big\{C_5J_5(\eta) + C_6J_6(\eta) + C_7J_7(\eta) + C_8J_8(\eta)\Big\}\Bigg] \qquad (5.5.27)$$

where J_5, J_6, J_7 and J_8 are defined by Eqs. (5.5.23) and (5.5.24). The auxiliary equations (5.5.25) and (5.5.26) hold in this case also.

5.6 Application of Boundary Conditions

The boundary conditions (5.2.6) - (5.2.20) involve derivatives of ψ_1, ψ_2, θ_1, θ_2 and χ. These differentiations necessitate the use of Leibniz's rule since the variables ξ and η occur in the limits of integration. Appendix F contains the expressions for the required derivatives.

Substituting Eqs. (D.1) - (D.7), (D.11) and (D.16) into the boundary conditions (5.2.6) - (5.2.10) and carrying out the necessary algebraic simplifications, the resulting equations are

$$\alpha_1 e^{-\alpha_1} C_1 + \alpha_1 e^{\alpha_1} C_2 + I_1 C_3 + I_2 C_4 = 0 \tag{5.6.1}$$

$$\alpha_1 e^{-\alpha_1} C_1 - \alpha e^{\alpha_1} C_2 + I_3 C_3 + I_4 C_4 = 0 \tag{5.6.2}$$

$$\alpha_2 e^{\alpha_2} C_5 + \alpha_2 e^{-\alpha_2} C_6 + I_5 C_7 + I_6 C_8 = 0 \tag{5.6.3}$$

$$\alpha_2 e^{\alpha_2} C_5 - \alpha_2 e^{-\alpha_2} C_6 + I_7 C_7 + I_7 C_8 = 0 \tag{5.6.4}$$

$$\begin{aligned} &\overline{u}\alpha_1 C_1 - \overline{u}\alpha_1 C_2 + \overline{u} I_{11} C_3 + \overline{u} I_{12} C_4 - \varepsilon\alpha_2 C_5 + \varepsilon\alpha_2 C_6 - \varepsilon I_{15} C_7 - \varepsilon I_{16} C_8 \\ &= i\alpha_1(\overline{u}\hat{u}_1' - \varepsilon\dot{\hat{u}}_2) + \alpha_1\left(\alpha_2\varepsilon\frac{R_\delta}{R_2} - \alpha_1\overline{u}\,\frac{R_h}{R_1}\right) \end{aligned} \tag{5.6.5}$$

$$\begin{aligned} &2\overline{u}\alpha_1^2 C_1 + 2\overline{u}\alpha_1^2 C_2 + \left\{\overline{u}Ai\{\zeta_1(0)E^+\} - 2\alpha_1 I_9\right\} C_3 \\ &+ \left\{\overline{u}Ai\{\zeta_1(0)E^-\} - 2\alpha_1 I_{10}\right\} C_4 - 2\overline{\mu}\varepsilon^2 C_5 - 2\overline{\mu}\varepsilon^2 C_6 \\ &+ \left\{-\overline{\mu}\varepsilon^2 Ai\{\zeta_2(0)E^+\} + 2\overline{\mu}\varepsilon^2\alpha_2 I_{13}\right\} C_7 + \left\{-\overline{\mu}\varepsilon^2 Ai\{\zeta_2(0)E^-\} + 2\overline{\mu}\varepsilon^2\alpha_2 I_{14}\right\} C_8 \\ &= 0 \end{aligned} \tag{5.6.6}$$

$$
\left[- \frac{\overline{u}^2}{\overline{\overline{\rho}}} \left\{ \frac{R_h}{R_1} + \frac{i}{\alpha_1} \left(\hat{u}_1' - (\hat{u}_1(0) - c_1)\alpha_1 \right) \right\} + \frac{2\overline{u}\alpha_1}{\varepsilon\overline{\mu}R_2} \right] C_1
$$

$$
+ \left[- \frac{\overline{u}^2}{\overline{\overline{\rho}}} \left\{ \frac{R_h}{R_1} + \frac{i}{\alpha_1} \left(\hat{u}_1' + (\hat{u}_1(0) - c_1)\alpha_1 \right) \right\} - \frac{2\overline{u}\alpha_1}{\varepsilon\overline{\mu}R_2} \right] C_2
$$

$$
+ \Bigg[- \frac{\overline{u}^2}{\overline{\overline{\rho}}} \frac{1}{\alpha_1^2} \left\{ - \frac{1}{2} \frac{R_h}{R_1} Ai\{\zeta_1(0)E^+\} + \frac{1}{R_1} Ai'\{\zeta_1(0)E^+\}\zeta_1' E^+ \right\}
$$

$$
+ \frac{\overline{u}^2}{\overline{\overline{\rho}}} \left\{ \frac{1}{\alpha_1} \frac{R_h}{R_1} + \frac{i\hat{u}_1'}{\alpha_1^2} \right\} I_9 + \left\{ \frac{i}{\alpha_1} \frac{\overline{u}^2}{\overline{\overline{\rho}}} (\hat{u}_1(0) - c_1) + \frac{2\overline{u}}{\varepsilon\overline{\mu}R_2} \right\} I_{11} \Bigg] C_3
$$

$$
+ \Bigg[- \frac{\overline{u}^2}{\overline{\overline{\rho}}} \frac{1}{\alpha_1^2} \left\{ - \frac{1}{2} \frac{R_h}{R_1} Ai\{\zeta_1(0)E^-\} + \frac{1}{R_1} Ai'\{\zeta_1(0)E^+\}\zeta_1' E^- \right\}
$$

$$
+ \frac{\overline{u}^2}{\overline{\overline{\rho}}} \left\{ \frac{1}{\alpha} \frac{R_h}{R_1} + \frac{i\hat{u}_1}{\alpha_1^2} \right\} I_{10} + \left\{ \frac{i}{\alpha_1} \frac{\overline{u}^2}{\overline{\overline{\rho}}} (\hat{u}_1(0) - c_1) + \frac{2\overline{u}}{\varepsilon\overline{\mu}R_2} \right\} I_{12} \Bigg] C_4
$$

$$
+ \left[\frac{R_\delta}{R_2} + \frac{i}{\alpha_2} \left(\dot{\hat{u}}_2 - (\hat{u}_2(0) - c_2)\alpha_2 \right) - \frac{2\alpha_2}{R_2} \right] C_5
$$

$$
+ \left[\frac{R_\delta}{R_2} + \frac{i}{\alpha_2} \left(\dot{\hat{u}}_2 + (\hat{u}_2(0) - c_2)\alpha_2 \right) + \frac{2\alpha_2}{R_2} \right] C_6
$$

$$
+ \Bigg[\frac{1}{\alpha_2^2} \left\{ - \frac{1}{2} \frac{R_\delta}{R_2} Ai\{\zeta_2(0)E^+\} + \frac{1}{R_2} Ai'\{\zeta_2(0)E^+\}\dot{\zeta}_2 E^+ \right\}
$$

$$
- \left\{ \frac{1}{\alpha_2} \frac{R_\delta}{R_2} + \frac{i\dot{\hat{u}}_2}{\alpha_2^2} \right\} I_{13} - \left\{ \frac{i}{\alpha_2} (\hat{u}_2(0) - c_2) + \frac{2}{R_2} \right\} I_{15} \Bigg] C_7
$$

$$
+ \Bigg[\frac{1}{\alpha_2^2} \left\{ - \frac{1}{2} \frac{R_\delta}{R_2} Ai\{\zeta_2(0)E^-\} + \frac{1}{R_2} Ai'\{\zeta_2(0)E^-\}\dot{\zeta}_2 E^- \right\}
$$

$$
- \left\{ \frac{1}{\alpha_2} \frac{R_\delta}{R_2} + \frac{i\dot{\hat{u}}_2}{\alpha_2^2} \right\} I_{14} - \left\{ \frac{i}{\alpha_2} (\hat{u}_2(0) - c_2) + \frac{2}{R_2} \right\} I_{16} \Bigg] C_8 = - \frac{\overline{u}^2}{\overline{\overline{\rho}}} (\alpha^2 W^2 + \frac{1}{F^2}) \tag{5.6.7}
$$

$$\alpha_1\alpha_2\overline{u}C_1 + \alpha_1\alpha_2\overline{u}C_2 - \alpha_2\overline{u}I_9C_3 - \alpha_2\overline{u}I_{10}C_4$$

$$- \alpha_1\alpha_2\overline{\rho}C_5 - \alpha_1\alpha_2\overline{\rho}C_6 + \alpha_1\overline{\rho}I_{13}C_7 + \alpha_1\overline{\rho}I_{14}C_8$$

$$= i\alpha_1^2\alpha_2\left[\overline{u}(\hat{u}_1(0) - c_1) - \overline{\rho}(\hat{u}_2(0) - c_2)\right] \tag{5.6.8}$$

$$\left[R_2(1 - \hat{\chi}(0)) + P_2\{\dot{z}_2I_{31} - \frac{R_\delta}{2} I_{29}\}\right] C_5$$

$$+ \left[R_2(1 - \hat{\chi}(0)) + P_2\{\dot{z}_2I_{32} - \frac{R_\delta}{2} I_{30}\}\right] C_6$$

$$+ \left[- \frac{R_2}{\alpha_2}(1 - \hat{\chi}(0))I_{13} + P_2\{\dot{z}_2I_{47} - \frac{R_\delta}{2}I_{45}\}\right] C_7$$

$$+ \left[- \frac{R_2}{\alpha_2}(1 - \hat{\chi}(0))I_{14} + P_2\{\dot{z}_2I_{48} - \frac{R_\delta}{2}I_{46}\}\right] C_8$$

$$+ \left[- \frac{R_\delta}{2}Ai\{z_2(0)E^+\} + \dot{z}_2Ai'\{z_2(0)E^+\}E^+\right] C_{13}$$

$$+ \left[- \frac{R_\delta}{2}Ai\{z_2(0)E^-\} + \dot{z}_2Ai'\{z_2(0)E^-\}E^-\right] C_{14}$$

$$= i\alpha_1R_2(1 - \hat{\chi}(0))(\hat{u}_2(0) - c_2) \tag{5.6.9}$$

where

$$P_2 = \frac{2\pi R_2\dot{\hat{\chi}}}{(\alpha_2R_2\dot{\hat{u}}_2)^{1/3}}$$

$$Q_1I_{17}C_1 + Q_1I_{18}C_2 + Q_1I_{33}C_3 + Q_1I_{34}C_4$$

$$+ Ai\{z_1(-1)E^+\}C_9 + Ai\{z_1(-1)E^-\}C_{10} = 0 \tag{5.6.10}$$

where

$$Q_1 = \frac{2\pi Pr_1 R_1 \hat{T}_1'}{(\alpha_1 Pr_1 R_1 \hat{u}_1')^{1/3}}$$

$$Q_2 I_{21} C_5 + Q_2 I_{22} C_6 + Q_2 I_{37} C_7 + Q_2 I_{38} C_8$$

$$+ Ai\{z_2(1)E^+\}C_{11} + Ai\{z_2(1)E^-\}C_{12} = 0 \qquad (5.6.11)$$

where

$$Q_2 = \frac{2\pi R_2 \dot{\hat{T}}_2}{(\alpha_2 R_2 \hat{\dot{u}}_2)^{1/3}}$$

$$Q_1 \overline{T} I_{25} C_1 + Q_1 \overline{T} I_{26} C_2 + Q_1 \overline{T} I_{41} C_3 + Q_1 \overline{T} I_{42} C_4$$

$$-Q_2 I_{29} C_5 - Q_2 I_{30} C_6 - Q_2 I_{45} C_7 - Q_2 I_{46} C_8$$

$$+ \overline{T} Ai\{z_1(0)E^+\}C_9 + \overline{T} Ai\{z_1(0)E^-\}C_{10}$$

$$- Ai\{z_2(0)E^+\}C_{11} - Ai\{z_2(0)E^-\}C_{12} = \varepsilon \dot{\hat{T}}_2 - \overline{T}\hat{T}_1' \qquad (5.6.12)$$

$$-\left[\frac{\overline{T}Q_1}{\varepsilon\overline{k}}\left\{z_1' I_{27} + \frac{R_h Pr_1}{2} I_{25}\right\}\right] C_1 + \left[-\frac{\overline{T}Q_1}{\varepsilon\overline{k}}\left\{z_1' I_{28} + \frac{R_h Pr_1}{2} I_{26}\right\}\right] C_2$$

$$+ -\left[\frac{\overline{T}Q_1}{\varepsilon\overline{k}}\left\{z_1' I_{43} + \frac{R_h Pr_1}{2} I_{41}\right\}\right] C_3 + \left[-\frac{\overline{T}Q_1}{\varepsilon\overline{k}}\left\{z_1' I_{44} + \frac{R_h Pr_1}{2} I_{42}\right\}\right] C_4$$

$$+ \left[Q_2\left\{\dot{z}_2 I_{31} + \frac{R_\delta}{2} I_{29}\right\} - \frac{R_2\overline{T}\Lambda}{\overline{c}_p}\right] C_5$$

$$+ \left[Q_2\left\{\dot{z}_2 I_{32} + \frac{R_\delta}{2} I_{30}\right\} - \frac{R_2\overline{T}\Lambda}{\overline{c}_p}\right] C_6$$

$$+ \left[Q_2 \left\{ \dot{z}_2 I_{47} + \frac{R_\delta}{2} I_{45} \right\} + \frac{R_2 \overline{T} \Lambda}{\alpha_2 \overline{c}_p} I_{13} \right] C_7$$

$$+ \left[Q_2 \left\{ \dot{z}_2 I_{48} + \frac{R_\delta}{2} I_{46} \right\} + \frac{R_2 \overline{T} \Lambda}{\alpha_2 \overline{c}_p} I_{14} \right] C_8$$

$$+ \left[- \frac{\overline{T}}{\varepsilon \overline{k}} \left\{ Ai'\{z_1(0)E^+\} z_1' E^+ + \frac{R_h Pr_1}{2} Ai\{z_1(0)E^+\} \right\} \right] C_9$$

$$+ \left[- \frac{\overline{T}}{\varepsilon \overline{k}} \left\{ Ai'\{z_1(0)E^-\} z_1' E^- + \frac{R_h Pr_1}{2} Ai\{z_1(0)E^-\} \right\} \right] C_{10}$$

$$+ \left[Ai'\{z_2(0)E^+\} \dot{z}_2 E^+ + \frac{R_\delta}{2} Ai\{z_2(0)E^+\} \right] C_{11}$$

$$+ \left[Ai'\{z_2(0)E^-\} \dot{z}_2 E^- + \frac{R_\delta}{2} Ai\{z_2(0)E^-\} \right] C_{12}$$

$$= - \frac{R_2 \overline{T} \Lambda}{\overline{c}_p} i\alpha_1 (\hat{u}_2(0) - c_2) \tag{5.6.13}$$

$$P_2 I_{21} C_5 + P_2 I_{22} C_6 + P_2 I_{37} C_7 + P_2 I_{38} C_8$$

$$+ Ai\{z_2(1)E^+\} C_{13} + Ai\{z_2(1)E^-\} C_{14} = 0 \tag{5.6.14}$$

$$\left[- \frac{1}{E} \left\{ \frac{R_\delta}{R_2} - \frac{i}{\alpha_2} \left(\dot{\hat{u}}_2 - \alpha_2 (\hat{u}_2(0) - c_2) \right) \right\} + \frac{Q_2 Y_2 I_{29}}{\dot{\hat{T}}_2} \right] C_5$$

$$\left[- \frac{1}{E} \left\{ \frac{R_\delta}{R_2} - \frac{i}{\alpha_2} \left(\dot{\hat{u}}_2 + \alpha_2 (\hat{u}_2(0) - c_2) \right) \right\} + \frac{Q_2 Y_2 I_{30}}{\dot{\hat{T}}_2} \right] C_6$$

$$+ \left[\frac{1}{E\alpha_2^2} \left\{ - \frac{1}{2} \frac{R_\delta}{R_h} Ai\{\zeta_2(0)E^+\} + \frac{1}{R_2} Ai'\{\zeta_2(0)E^+\} \dot{\zeta}_2 E^+ \right. \right.$$

$$\left. \left. + \alpha_2 \left(\frac{R_\delta}{R_2} - \frac{i\dot{\hat{u}}_2}{\alpha_2} \right) I_{13} - i\alpha_2 (\hat{u}_2(0) - c_2) I_{15} \right\} + \frac{Q_2 Y_2 I_{45}}{\dot{\hat{T}}_2} \right] C_7$$

$$+\left[\frac{1}{E\alpha_2^2}\left\{-\frac{1}{2}\frac{R_\delta}{R_2}\,\mathrm{Ai}\{\zeta_2(0)E^-\} + \frac{1}{R_2}\,\mathrm{Ai}'\{\zeta_2(0)E^-\}\dot{\zeta}_2E^-\right.\right.$$

$$\left.\left. + \alpha_2\left(\frac{R_\delta}{R_2} - \frac{i\dot{\hat{u}}_2}{\alpha_2}\right) I_{14} - i\alpha_2(\hat{u}_2(0) - c_2)I_{16}\right\} + \frac{Q_2Y_2I_{46}}{\dot{\hat{T}}_2}\right] C_8$$

$$+\left[-\frac{\bar{\bar{R}}T}{\{\hat{T}_2(0)\}^2}\mathrm{Ai}\{z_2(0)E^+\}\right] C_{11} + -\frac{\bar{\bar{R}}T}{\{\hat{T}_2(0)\}^2}\,\mathrm{Ai}\{z_2(0)E^-\}\quad C_{12}$$

$$+\left[\frac{1}{\hat{\chi}(0)}\,\mathrm{Ai}\{z_2(0)E^+\}\right] C_{13} + \left[\frac{1}{\hat{\chi}(0)}\,\mathrm{Ai}\{z_2(0)E^-\}\right] C_{14} = 0 \qquad (5.6.15)$$

where

$$Y_2 = \frac{\dot{\hat{\chi}}}{\hat{\chi}(0)} - \frac{\bar{\bar{R}}T\dot{\hat{T}}_2}{\{\hat{T}_2(0)\}^2}$$

The definitions of integrals I_1 - I_{48} are listed in Appendix E.

As in the case of zero mass transfer the boundary conditions (5.6.1) - (5.6.7) and (5.6.9) - (5.6.15) form a system of linear algebraic equations of the type

$$[A(c_1)]\{c\} = \{V(c_1)\} \qquad (5.6.16)$$

where $[A]$ is a 14 x 14 coefficient matrix of the left hand sides and $\{V(c_1)\}$ is a 14 x 1 column vector of the right hand sides. The remaining equation represents the characteristic function for the mass transfer problem and can be written in the form

$$G\left[c_1;\{C(c_1)\}\right] = \alpha_2\bar{u}\left[\alpha_1C_1 + \alpha_1C_2 - I_9C_3 - I_{10}C_4\right]$$

$$-\alpha_1\bar{\rho}\left[\alpha_2C_5 + \alpha_2C_6 - I_{13}C_7 - I_{14}C_8\right]$$

$$-i\alpha_1^2\alpha_2\left[\,\bar{u}(\hat{u}_1(0) - c_1) - \bar{\rho}(\hat{u}_2(0) - c_2)\right] = 0$$

(5.6.17)

or in a condensed notation

$$G\left[c_1;\{C(c_1)\}\right] = \{C(c_1)\}^T\{U(c_1)\} - i\alpha_1^2\alpha_2\left[\,\bar{u}(\hat{u}_1(0) - c_1) - \bar{\rho}(\hat{u}_2(0) - c_2)\right] \tag{5.6.18}$$

where

$$\{C(c_1)\} = [A(c_1)]^{-1}\{V(c_1)\} \tag{5.6.19}$$

and

$$\{U(c_1)\}^T = \{\alpha_1\alpha_2\bar{u} \quad \alpha_1\alpha_2\bar{u} \quad -\alpha_2\bar{u}I_9 \quad -\alpha_2\bar{u}I_{10} \quad -\alpha_1\alpha_2\bar{\rho} \quad -\alpha_1\alpha_2\bar{\rho} \quad \alpha_1\bar{\rho}I_{13} \quad \alpha_1\bar{\rho}I_{14}\} \tag{5.6.20}$$

In Eqs. (5.6.17) and (5.6.18), c_2 is again given by Eq. (4.1.15). As in Chapter IV G is the characteristic function and c_1 is the eigenvalue. The function G can be expressed as

$$G(\alpha_1,\ \varepsilon, \bar{\mu},\ \bar{k},\ \bar{\rho},\ \bar{c}_p,\ R_2,\ W, F,\ E,\ R_h,\ R_\delta,\ Pr_1,\ \Lambda,\ \bar{\bar{R}};\ c_1) = 0 \tag{5.6.21}$$

The stability problem is therefore reduced to locating the zeros of an analytic function G in the complex c_1 plane. The method of zero finding is described in the next section.

5.7 Outline of the Eigenvalue Iteration Procedure

The zeros of the characteristic function G are determined numberically using methods similar to those of Sec. 4.5. The important steps are listed below.

(i) The initial guess for the eigenvalue c_1 is obtained from the solution of the zero mass transfer problem. This restricts the investigation to the effects of mass transfer on particular modes and the question whether mass transfer itself introduces any stability modes is left unanswered. The latter statement will become more meaningful in conjunction with the developments of Chapter VI.

(ii) For given parameter values in Eq. (5.6.21) and the guess value c_1, evaluate the integrals I_1 through I_{48} using the integration procedures described in Appendices E and G.

(iii) Generate the coefficient matrix $[A]$ and obtain its inverse. The IBM routine MINV was used for this purpose with minor modifications. Also generate the column vector $\{V\}$.

(iv) Determine the constants of integration $\{C\}$ by computing the product $[A]^{-1}\{V\}$.

(v) Calculate G in Eq. (5.6.17). Usually this equation will not be satisfied with the first guess for c_1.

(vi) Obtain an improved approximation for c_1 by employing the Newton-Raphson iteration technique described in Appendix I.

(vii) Compare successive values of c_1 for convergence within a prescribed tolerance on real and imaginary parts. Repeat steps (ii) - (vi) until desired convergence is reached.

CHAPTER VI

RESULTS AND DISCUSSION

6.1 Description of Data

The experimental data of Craik[22] was chosen for numerical computations. The present theoretical model does not match the experimental conditions perfectly and hence some of the parameters in Craik's investigation were suitably 'corrected'. For instance, the experiments were conducted in a closed rectangular channel 1 inch high and 6 inches wide. The air flow was provided by a large fan which drew air through the apparatus. Water was introduced into the channel at the entry section and formed a film on the bottom made of plate glass. Consequently the velocity profile in air was parabolic but the liquid velocity profile was very nearly linear. It was assumed in the present analysis that the velocity profile in the gas is linear and has a thickness δ. Therefore, in order to adapt Craik's data to the present model, δ was chosen to be half the difference between channel height and liquid film thickness. The velocity profile between the interface and $y = \delta$ was assumed linear.

Another important correction to Craik's data was made in regard to the value of gas viscosity. The air flow in his experiments was turbulent and hence an augmented value of laminar viscosity must be used. This is accomplished as follows. Craik measures the steady-state interface velocity u_{if} and determines the film thickness h by measuring the volumetric flow rate of liquid. This permits one to compute the interfacial shear on the liquid side using the expression

$$\tilde{\tau}_1 = \frac{\mu_1 u_{if}}{h} \tag{6.1.1}$$

Since this value of shear stress must equal the interfacial shear on the gas side, it follows that

$$\tilde{\tau}_1 = \tilde{\tau}_2 = \frac{\mu_2 u_e}{\delta} \tag{6.1.2}$$

and therefore,

$$\mu_2 = \frac{\delta}{h} \frac{u_{if}}{u_e} \mu_1 \tag{6.1.3}$$

Since the gas Prandtl number Pr_2 is assumed unity, the thermal conductivity k_2 of the gas for turbulent flow is calculated using the relation

$$k_2 = k_{2\ell am} F_k \tag{6.1.4}$$

where the factor F_k is given by

$$F_k = \frac{\mu_2}{\mu_{2\ell am}} \tag{6.1.5}$$

In Eqs. (6.1.4) and (6.1.5) $k_{2\ell am}$ and $\mu_{2\ell am}$ denote molecular thermal conductivity and viscosity respectively.

The list of various dimensional and non-dimensional parameters used in numerical computations is contained in Table II at the end of this chapter. The data in this Table corresponds to both air and water at room temperature.

6.2 Root Location Procedure for Zero Mass Transfer Problem

One of the eigenvalues (or modes) can be immediately calculated from the long wavelength disturbance solution of Sec. 4.2. Thus substituting for ε and μ from Table II into the approximate expression (4.2.19) and the exact expression (4.2.18) the results are

$$\left.\begin{array}{l} c_{1approx} = 1.086 + 0i \\ \\ c_{1exact} = 1.064 + 0i \end{array}\right. \quad \text{for } \alpha_1 << 1 \text{ and } \alpha_2 << 1$$

Starting with the guess $c_1 = 1.086$ and choosing $\alpha_1 = 0.001$ (hence $\alpha_2 = \alpha_1/\varepsilon = 0.023$) the Newton-Raphson procedure yields the result

$$c_1 = 1.06438 - 0.00774 \quad \text{for } \alpha_1 = 0.001$$

Once c_1 is known for a given α_1, it is a simple matter to trace this mode by varying α_1 gradually.

Identification of other stability modes is somewhat complicated and tedious. The reader is referred to Sec. 4.6 in this connection. An illustrative example of approximate location of roots of the characteristic equation (4.4.11) is given in Figs. 5 and 6. Suppose it is desired to find the eigenvalues (i.e. roots of the characteristic equation) in the interval $1 \leq c_{1r} \leq 2$ and $-1 \leq c_{1i} \leq 0$. Then the function G in Eq. (4.4.16) is calculated at $c_{1i} = 0, -0.1, -0.2, \ldots, -1.0$ for each $c_{1r} = 1.0, 1.1, \ldots, 2.0$. The next step is to plot Re(G) and Im(G) against c_{1r} with c_{1i} as a parameter. The results are shown in Figs. 5 and 6 for the case $\alpha_1 = 0.05$. This choice of α_1 was dictated by the fact that the system is expected to be well-behaved for long wavelength disturbances. It is seen that both Re(G) and Im(G) go to zero around $c_1 = 1.32 - 0.4i$ and $c_1 = 1.84 - 0.5i$. With these guess values the Newton-Raphson iteration gives the exact results

$$\left.\begin{array}{l} c_1 = 1.318 - 0.405i \\ \text{and} \\ c_1 = 1.842 - 0.505i \end{array}\right\} \text{for } \alpha_1 = 0.05$$

This process of root location was carried out in the interval $0 \leq c_{1r} \leq 4$ and $-2 \leq c_{1i} < 2$. The following spectrum of eigenvalues, ordered according to real part, was obtained:

For $\alpha_1 = 0.05$,

$$c_{11} = 0.273 - 0.197i$$

$$c_{12} = 1.318 - 0.405i$$

$$c_{13} = 1.411 + 1.053i$$

$$c_{14} = 1.842 - 0.505i$$

$$c_{15} = 3.263 - 1.220i$$

Note that only c_{11} has a critical point inside the liquid (i.e., $c_{11r} < 1$.) A random search for an eigenvalue with $c_{1r} > 4$ led to the root

$$c_{16} = 17.48 + 0.540i$$

It should be emphasized that this eigenvalue was obtained by supplying random guesses (with $c_{1r} > 4$) and was not obtained through the systematic search procedure mentioned earlier.

6.3 Amplification and Phase Velocity Curves for Zero Mass Transfer Case

It was stated in Chapter I that the stability or instability of the interface depends upon whether an infinitesimal disturbance of a given wavelength grows or decays with time. As pointed out in Sec. 2.7 positive ω_i (or c_i) corresponds to an unstable interface and negative ω_i corresponds to a stable interface. With these comments in mind one seeks to know how ω_i (or more correctly $\alpha_1 c_{1i}$) varies with the disturbance wave number $k(= 2\pi/\lambda)$. This requires that the modes obtained in Sec. 6.2 for $\alpha_1 = 0.05$ be traced as α_1 changes continuously. Each of the six modes mentioned in the previous section was traced by varying α_1 very gradually. Much care needs to be exercised during this process since a sufficiently large change in α_1 may result in switching to a different mode. This exercise was carefully done and the results are presented in Figs. 6 through 11 as amplification and phase velocity curves. The plots of phase velocity against wave number have been included for the sake of completeness and also to facilitate comparisons with the well-known water wave phenomena such as gravity-surface tension waves and Kelvin-Helmholtz waves. The curves of amplification have been presented in the form $\alpha_1 c_{1i}$ vs α_1 and the phase velocities are plotted in the form $\alpha_1 c_{1r}$ vs α_1. It may be recalled that $\alpha_1 c_{1i}$ and $\alpha_1 c_{1r}$ are proportional to ω_i and ω_r respectively.

A very important point needs to be brought to the attention of the reader. Only those modes discovered for $\alpha_1 = 0.05$ in the previous section have been traced as α_1 varies. It is conceivable that more stability modes 'creep in' as α_1 increases. In fact, some evidence was obtained during the numerical investigation which showed that this is true. It should soon become clear that the six modes in the present investigations have rather distinct characteristics. These are discussed below.

(i) Figs. 6a and 6b represent amplification and phase velocity characteristics for the eigenvalue c_{11}. This eigenvalue is of some interest because it is the only one with a critical point inside the liquid. Fig. 6a shows that the imaginary part of c_{11} is always less than zero and hence this mode is stable for all α_1. The curious dip around $\alpha_1 = 0.65$ is not explained. The phase velocity plot of Fig. 6b also exhibits sharp changes at this value of α_1. The phase velocity is generally increasing with α_1.

(ii) The eigenvalue c_{12} is interesting because it was the only one predicted analytically. It may be recalled from Sec. 4.2 that this mode (at least for small values of α_1) has a physical interpretation that it depends only on the thickness ratio ε and viscosity ratio $\overline{\mu}$, and it is independent of Reynolds, Froude and Weber numbers. The amplification curve in Fig. 7a shows that this eigenvalue is also stable for all α_1 and displays a small dip around $\alpha_1 = 0.6$. The foregoing observations suggest that this eigenvalue can be associated with the Tollmien-Schlichting mode of stability (Sec. 1.2i). The phase velocity (Fig. 7b) increases continuously with α_1 but at a much faster rate than the c_{11} mode.

(iii) The eigenvalue c_{13} was found to be the only unstable mode at $\alpha_1 = 0.05$ and deserves attention for this reason. The variation of amplification rate (Fig. 8a) shows that this mode is unstable for all α_1 (except at $\alpha_1 = 0$ where it is neutrally stable). It is seen that the rate of amplification increases rapidly beyond $\alpha_1 \simeq 0.5$. The phase velocity plot of Fig. 8b also displays a sharp change around this value of α_1. The phase velocity increases with α_1 in this case also.

(iv) Fig. 9a represents amplification curve for the eigenvalue c_{14}. It exhibits a unique characteristic in that this mode is stable at small values of α_1 and unstable at large α_1. In other words, there are two distinct regions, one stable and the other unstable, separated by a neutral stability point. With the exception of c_{16} (which will be discussed shortly) none of the other modes demonstrate this behavior. It appears that this mode is the same as the one obtained by Bordner, et. al[37] using the data of Cohen and Hanratty[30]. It is interesting to compare the phase velocity for this eigenvalue with the speed of propagation of surface tension - gravity waves and Kelvin-Helmholtz waves (Fig. 9b). It is observed that long wavelength disturbances (i.e., small α_1) of this mode propagate with nearly the same speed as Kelvin-Helmholtz waves. The mode under consideration can be associated with the class C (or Kelvin-Helmholtz) instability of Benjamin[9] and Landahl[10]. This eigenvalue, therefore, is called the modified Kelvin-Helmholtz mode in the present investigation. The expressions used for calculating the speeds of surface tension-gravity waves and Kelvin-Helmholtz waves, in dimensionless form, are given below:

$$c_o^2 = \frac{1}{\alpha_1 F^2}\left[\frac{1 - \bar{\rho} + \alpha_1^2 W^2 F^2}{\coth(\alpha_1) + \bar{\rho}\coth(\alpha_2)}\right] \quad \text{surface tension-gravity wave} \tag{6.3.1}$$

$$c_o = \frac{\coth(\alpha_1) + \frac{\bar{\rho}}{\bar{u}}\coth(\alpha_2)}{\coth(\alpha_1) + \bar{\rho}\coth(\alpha_2)} + \left[\frac{1}{\alpha_1 F^2}\left\{\frac{1 - \bar{\rho} + \alpha_1^2 W^2 F^2}{\coth(\alpha_1) + \bar{\rho}\coth(\alpha_2)}\right\} - \frac{\bar{\rho}(1 - \frac{1}{\bar{u}})^2 \coth(\alpha_1)\coth(\alpha_2)}{\{\coth(\alpha_1) + \bar{\rho}\coth(\alpha_2)\}^2}\right]^{1/2}$$

Kelvin-Helmholtz wave (6.3.2)

These equations hold for two fluids bounded between two walls at $y = -h$ and $y = \delta$ respectively.

Finally, it is observed from Fig. 9a that instability sets in when α_1 is above 0.145. Corresponding to this value of α_1, $c_{1r} = 1.99$ from Fig. 9b. The dimensional value of c_{1r} is then 13.5 cm/sec and compares favorably with Craik's experimental value of 11.9 cm/sec at the onset of instability.

(v) The amplification and phase velocity curves corresponding to the eigenvalue c_{15} are plotted in Figs. 10a and 10b respectively. The phase velocity increases with α_1 similar to the other modes and complete stability prevails for all α_1 as shown by the amplification plot. Thus this mode does not display any peculiar characteristics.

(vi) The eigenvalue c_{16} is a representative fast moving disturbance and therefore merits some consideration. Fig. 11a shows that the amplification curve oscillates rapidly with α_1 and is somewhat irregular. Thus there are several regions of stability and instability separated by neutral stability points. The phase velocity curve (Fig. 11b) also exhibits irregular behavior. In fact, the phase velocity decreases with α_1 over a small range.

The above description of six different stability modes shows that amongst the slow moving disturbances the modified Kelvin-Helmholtz mode is the most interesting. Therefore, further attention is concentrated on this particular mode.

6.4 Effect of Neglecting Instability in Gas

It was mentioned in Secs. 1.2 and 2.1(2) that the validity of the frequently-made assumption of neglecting the phase speed in gas disturbance equations is doubtful. To examine this question the zero mass transfer problem was solved putting $c_2 = 0$ in gas disturbance equations (4.1.2), (4.1.9) and (4.1.11). This amounts to assuming the disturbance in Eq. (2.6.7) to be of the form

$$q_{21}(x,y,t) = q_2(y)e^{ikx} \qquad (6.4.1)$$

This exercise required only minor modifications in the Newton-Raphson procedure and in the computer program. As pointed out in the previous section, only the modified Kelvin-Helmholtz mode was considered. The results of these computations are shown in Figs. 12a and 12b, again in the form of amplification and phase velocity curves. The amplification plot for $c_2 \neq 0$ is a portion of Fig. 9a. A comparison of the curves for $c_2 \neq 0$ and $c_2 = 0$ reveals the following significant results.

(i) The assumption of neglecting the instability in gas has no effect at low wave numbers (i.e. for disturbances with long wavelengths) and affects the neutrally stable wave number only slightly.

(ii) Beyond the neutrally stable wave number the above assumption results in underestimation of the amplification rate.

(iii) When the wave number is sufficiently large (e.g. $\alpha > 0.4$ in Fig. 12a) the $c_2 = 0$ assumption predicts stability when the interface is actually unstable.

Fig. 12b shows that there is no appreciable difference between phase velocity curves for the cases $c_2 \neq 0$ and $c_2 = 0$. Another important aspect of the assumption under question needs to be investigated. It was pointed out in Secs. 1.2 and 2.1(2) that Benjamin[21] established the criterion which allows one to make the rigid wavy wall (or $c_2 = 0$) assumption. This suggests calculation of Benjamin's parameter in Eq. (2.1.1) for different values of α_1. This parameter written for the gas (i.e. fluid '2') is

$$B_p = \frac{m_2 c_{2r}}{\hat{\dot{u}}_2} << 1 \qquad (6.4.2)$$

with

$$m_2 = \left\{\alpha_2 R_2 \dot{\hat{u}}\right\}^{2/3} \tag{6.4.3}$$

Fig. 13 shows a plot of B_p against α_1. It is seen that $B_p << 1$ holds only for very small values of α_1 (typically $\alpha_1 < 0.1$). When $\alpha_1 > 0.2$ Benjamin's parameter can no longer be considered small compared to unity. This discussion explains why the amplification curves for $c_2 \neq 0$ and $c_2 = 0$ display characteristically different behavior when $\alpha_1 > 0.1$.

6.5 Effects of Interface Mass Transfer

A typical example of the effects of interface evaporation on the modified Kelvin-Helmholtz mode is shown in Figs. 14a and 14b. Craik's data in Table II have been used in these computations. It should be noted that the evaporative mass transfer occurs at room temperature (68 F) and consequently the mass transfer Reynolds numbers R_h and R_δ are very small. The amplification and phase velocity curves with and without mass transfer are included in Figs. 14a and b. It is observed that both amplification curves coincide in the stable range and the neutrally stable wave number is very slightly affected. When α_1 is beyond the neutrally stable value, however, interface mass transfer results in an increase in amplification rate. It can therefore be concluded that the effect of mass transfer is destabilizing for this particular stability mode.

A comparison of phase velocity curves in Fig. 14b shows that mass transfer leads to a small increase in the phase velocity. In both Figs. 14a & b, the mass transfer effects seem to become more significant as α_1 increases. This is understandable since a highly rippled interface (α_1 large) causes increased mass transfer perturbations.

The mass transfer curves are terminated at $\alpha_1 = 0.25$ because the Newton-Raphson procedure failed to converge beyond this point. Further solutions were attempted using a conjugate gradient method (a variation of the method of steepest descent). This method is extremely slow because it calls the mass transfer program a large number of times. The workability of this method was tested for $\alpha_1 < 0.25$ during the last stages of the present investigation. Therefore, eigenvalue solutions for $\alpha_1 > 0.25$ could not be included in the present report.

6.6 Suggestions for Future Investigations

The experience gained in the present investigation suggests that the following areas of the stability problem need further study.

(i) The different modes of stability should be classified according to their physical interpretations. The works of Benjamin[9] and Landahl[10] should serve as models in this quest. It appears that a stability mode associated with a particular physical mechanism could be 'singled out' from the rest by employing some approximate technique. An insight into the physical phenomena is necessary to understand which modes occur in practice.

(ii) A combination of physical and mathematical reasoning is required to understand the structure of the entire eigenvalue spectrum. Recent investigations of Mack[13] show that the eigenvalue spectrum of a laminar boundary layer, with a discontinuous first derivative in velocity, is infinite.

(iii) The work on the liquid film stability problem must eventually include the effects of mean velocity profile curvature. Miles'[2] work shows that velocity profile curvature plays an important role in the energy transfer mechanism. Since the Orr-Sommerfeld equation cannot be solved exactly for a general velocity profile, a finite difference approach will have to be adopted. Therefore, one may consider solving the present linear velocity profile problem using finite differences as a first step. This exercise will help in gaining familiarity with the problems of eigenvalue location.

(iv) The stability problem with mass transfer needs to be studied in greater detail, even for an incompressible air flow. For instance, the effect of mass transfer on different stability modes should be investigated. Further numerical investigations should be carried out with the present model for higher rates of mass transfer. An important aspect of this problem, not covered in the present work, is whether the physical process of mass transfer itself introduces any modes of instability.

(v) Finally, one wishes to solve the problem of compressible boundary layer over a liquid film. The processes of heat and mass transfer will undoubtedly be extremely important, especially when the air flow is supersonic. The latter case is interesting because multiple stability loops are known to exist when the mean flow in the boundary layer becomes supersonic.

TABLE II

LIQUID PROPERTIES AT 528 DEG R

Liquid = Water

Molecular Weight = 18.0

Latent Heat of Vaporization = 971.65 Btu/lbm.

Coefficient of Viscosity = 2.107×10^{-5} lbm-sec/ft^2.

Density = 1.937 slug/ft^3.

Specific Heat = 1.00 Btu/lbm-deg R.

Thermal Conductivity = 9.611×10^{-5} Btu/ft-sec-deg R.

Surface Tension = 4.926×10^{-3} lbf/ft.

Liquid Layer Thickness = 1.755×10^{-3} ft.

Wall Temperature = 528.0 deg R.

GAS PROPERTIES AT 528 DEG R

Gas = Air

Coefficient of Viscosity = 5.549×10^{-6} lbf/ft^2.

Density = 2.340×10^{-3} slug/ft^3.

Specific Heat = 0.24 Btu/lbm-deg R.

Thermal Conductivity = 6.100×10^{-5} Btu/ft-sec-deg R.

Velocity at Edge of Boundary Layer = 19.66 ft/sec.

Static Pressure = 2.116×10^3 lbf/ft^2.

Temperature at Edge of Boundary Layer = 528.0 deg R.

Boundary Layer Thickness = 4.079×10^{-2} ft.

CHARACTERISTIC NON-DIMENSIONAL PARAMETERS

Liquid Layer/Boundary Layer Thickness: $\varepsilon = 4.302 \times 10^{-2}$

Gas Viscosity/Liquid Viscosity: $\overline{\mu} = 0.263$

Gas Density/Liquid Density: $\overline{\rho} = 1.208 \times 10^{-3}$

Gas Specific Heat/Liquid Specific Heat: $\overline{c}_p = 0.240$

Gas Thermal Conductivity/Liquid Thermal Conductivity: $\overline{k} = 0.635$

Liquid Reynolds Number: $R_1 = 35.54$

Gas Reynolds Number: $R_2 = 338.2$

Mass Transfer Reynolds Number for Liquid: $R_h = 2.95 \times 10^{-4}$

Mass Transfer Reynolds Number for Gas: $R_\delta = 2.60 \times 10^{-2}$

Liquid Prandtl Number: $Pr_1 = 10.04$

Gas Prandtl Number: $Pr_2 = 1.0$

Liquid Weber Number: $W = 5.46$

Liquid Froude Number: $F = 0.93$

Gas Euler Number: $E = 2.34 \times 10^3$

CHAPTER VII

CONCLUSIONS

The following general conclusions can be drawn from the present analysis of liquid film stability.

(i) The stability of an interface between two fluids is characterized by the existence of several modes of stability. These modes may be completely stable, unstable or may change from stable to unstable (or vice versa) as the disturbance wave number is varied. Two modes were distinguished physically in this investigation, one associated with the Tollmien-Schlichting instability and the other with Kelvin-Helmholtz instability. It should be pointed out that the foregoing observations hold for the case of linear velocity profiles in both the gas and the liquid. The assumption of linear profiles is introduced in order to isolate the effects of mass transfer on stability. Further research is recommended to study the eigenvalue spectrum for curved velocity profiles.

(ii) The customary assumption of neglecting instabilities in the incompressible gas motion is valid only for very small values of the disturbance wave number ($\alpha_1 << 1$). When the disturbance wave number is moderate, i.e., $\alpha_1 = O(1)$, such an assumption not only leads to a gross underestimation of the amplification rate (for the modified Kelvin-Helmholtz mode) but can even predict incorrectly a stable interface. Again, these conclusions apply to linear velocity profiles in both fluids and need to be extended to include the effects of velocity profile curvature.

(iii) Limited computations, for very small mass transfer rates, indicate that interface evaporation has a negligible effect on the modified Kelvin-Helmholtz stability mode for very small disturbance wave numbers and has a destabilizing effect when the wave number is moderate.

REFERENCES

1. Ursell, F., "Wave Generation by Wind," Surveys in Mechanics, Cambridge University Press, 1956, pp. 217-249.

2. Miles, J. W., "On the Generation of Surface Waves by Shear Flows," Journal of Fluid Mechanics, Vol. 3, pp. 185-204, 1957.

3. Miles, J. W., "On the Generation of Surface Waves by Shear Flows, Part 2," Journal of Fluid Mechanics, Vol. 6, pp. 568-582, 1959.

4. Phillips, O. M., "On the Generation of Waves by Turbulent Wind," Journal of Fluid Mechanics, Vol. 2, pp. 417-445, 1957.

5. Kinsman, B., Wind Waves, Prentice Hall, 1965, pp. 575-583.

6. Lighthill, M. J., "Physical Interpretation of the Mathematical Theory of Wave Generation by Wind," Journal of Fluid Mechanics, Vol. 14, pp. 385-398, 1962.

7. Lin, C. C., The Theory of Hydrodynamic Stability, Cambridge University Press, 1955.

8. Benjamin, T. B., "Effects of Flexible Boundary on Hydrodynamic Stability," Journal of Fluid Mechanics, Vol. 9, pp. 513-532, 1960.

9. Benjamin, T. B., "The Threefold Classification of Unstable Disturbances in Flexible Surfaces Bounding Inviscid Flows," Journal of Fluid Mechanics, Vol. 16, pp. 436-450, 1963.

10. Landahl, M. T., "On the Stability of a Laminar Incompressible Boundary Layer over a Flexible Surface," Journal of Fluid Mechanics, Vol. 13, pp. 609-632, 1962.

11. Skripachev, V. V., "Stability of the Laminar Boundary Layer on a Deformable Membrane, Supported on a Layer of an Inviscid, Incompressible Fluid," Fluid Mechanics-Soviet Research, Vol. 2, No. 5, pp. 21-26, Sept.-Oct. 1973.

12. Gaster, M. and Jordinson, R., "On the Eigenvalues of the Orr-Sommerfeld Equation," Journal of Fluid Mechanics, Vol. 72, pp. 121-133, 1975.

13. Mack, L. M., "A Numerical Study of the Temporal Eigenvalue Spectrum of the Blasius Boundary Layer," Journal of Fluid Mechanics, Vol. 73, pp. 497-520, 1976.

14. DiPrima, R. C. and Habetler, G. J., "A Completeness Theorem for Non-selfadjoint Eigenvalue Problems in Hydrodynamic Stability," Archives of Rational Mechanics and Analysis, Vol. 34, pp. 218-227, 1969.

15. Gallagher, A. P. and Mercer, A. McD., "On the Behavior of Small Disturbances in Plane Couette Flow," Journal of Fluid Mechanics, Vol. 13, pp. 91-100, 1962.

16. Gallagher, A. P. and Mercer, A. McD., "On the Behavior of Small Disturbances in Plane Couette Flow, Part 2," Journal of Fluid Mechanics, Vol. 18, pp. 350-352, 1964.

17. Deardorff, J. W., "On the Stability of Viscous Plane Couette Flow," Journal of Fluid Mechanics, Vol. 15, pp. 623-631, 1963.

18. Haines, F. D., "Stability of Plane Couette-Poiseuille Flow with Uniform Crossflow," The Physics of Fluids, Vol. 14, No. 8, pp. 1620-1623, Aug. 1971.

19. Lamb, H., Hydrodynamics, sixth edition, Dover Publications, 1945 Chapter IX.

20. Miles, J. W., "On the Generation of Surface Waves by Shear Flows, Part 3, Kelvin-Helmholtz Instability," Journal of Fluid Mechanics, Vol. 6, pp. 583-598, 1959.

21. Benjamin, T. B., "Shearing Flow Over a Wavy Boundary," Journal of Fluid Mechanics, Vol. 6, pp. 161-205, 1959.

22. Craik, A. D. D., "Wind Generated Waves in Thin Liquid Films," Journal of Fluid Mechanics, Vol. 26, pp. 369-392, 1966.

23. Lighthill, M. J., "On Boundary Layers and Upstream Influence II - Supersonic Flows without Separation," Proceedings of the Royal Society, Series A, Vol. 217, pp. 478-508, 1953.

24. Inger, G. R., "Compressible Boundary Layer Flow past a Swept Wavy Wall with Heat Transfer and Ablation," Astronautica Acta, Vol. 16, pp. 325-338, 1971.

25. Chandrasekhar, S., Hydrodynamic and Hydromagnetic Stability, Oxford University Press, 1961.

26. Nayfeh, A. H. and Saric, W. S., "Stability of a Liquid Film," AIAA Journal, Vol. 9, No. 4, pp. 750-752, April 1971.

27. Lock, R. C., "Hydrodynamic Stability of the Flow in the Laminar Boundary Layer between Parallel Streams," Proceedings of the Cambridge Philosophical Society, Vol. 50, pp. 105-124, 1954.

28. Feldman, S., "On the Hydrodynamic Stability of Two Viscous Incompressible Fluids in Parallel Uniform Shearing Motion," Journal of Fluid Mechanics, Vol. 2, pp. 343-370, 1957.

29. Miles, J. W., "The Hydrodynamic Stability of a Thin Film of Liquid in Uniform Shearing Motion," Journal of Fluid Mechanics, Vol. 8, pp. 593-610, 1960.

30. Cohen, L. S. and Hanratty, T. J., "Generation of Waves in the Concurrent Flow of Air and a Liquid," AIChE Journal, Vol. 31, pp. 138-144, 1965.

31. Plate, E. J., Chang, P. C. and Hidy, G. M., "Experiments on the Generation of Small Water Waves by Wind," Journal of Fluid Mechanics, Vol. 35, pp. 625-656, 1969.

32. Chang, I. D. and Russell, P. E., "Stability of a Liquid Layer Adjacent to a High-Speed Gas Stream," The Physics of Fluids, Vol. 8, No. 9, June, 1965.

33. Nachtscheim, P. R., "Analysis of the Stability of a Thin Liquid Film Adjacent to a High-Speed Gas Stream," NASA TN D-4976, Jan. 1969.

34. Saric, W. S. and Marshall, B. W., "An Experimental Investigation of the Stability of a Thin Liquid Layer Adjacent to a Supersonic Stream," AIAA Journal, Vol. 9, No. 8, pp. 1546-1553, Aug. 1971.

35. Starkenberg, J., "An Analysis of the Stability of Thin Liquid Films in Hypersonic Environments," PIBAL Report No. 72-11, Polytechnic Institute of Brooklyn, 1972.

36. Creski, R. J. and Starkenberg, J., "Liquid Film Cooling on Hypersonic Slender Bodies," Astronautica Acta, Vol. 18, pp. 11-20, 1973.

37. Bordner, G. L., Nayfeh, A. H. and Saric, W. S., "Stability of Liquid Films Adjacent to Compressible Streams," VPI Research Report VPI-E-73-3, Virginia Polytechnic Institute and State University, Jan. 1973.

38. Gater, R. A. and L'Ecuyer, M. R., "A Fundamental Investigation of the Phenomena that Characterize Liquid-Film Cooling," International Journal of Heat and Mass Transfer, Vol. 13, pp. 1925-1939, 1970.

39. Kotake, S., "Heat Transfer and Skin Friction of a Phase-Changing Interface of Gas-Liquid Laminar Flows," International Journal of Heat and Mass Transfer, Vol. 16, pp. 2165-2176, 1973.

40. Kotake, S., "Gas-Liquid Laminar Boundary-Layer Flows with a Wavy Phase-Changing Interface," International Journal of Heat and Mass Transfer, Vol. 17, pp. 885-897, 1974.

41. Kotake, S., "Gas-Liquid Laminar Boundary-Layer Flows with a Phase-Changing Interface and its Hydrodynamic Instability," Report No. 505(1974), Institute of Space and Aeronautical Science, University of Tokyo, Jan. 1974.

42. Nayfeh, A. H., Perturbation Methods, Wiley-Interscience Publication, 1973.

43. Karamcheti, K., Principles of Ideal Fluid Aerodynamics, John Wiley and Sons, Inc., 1966.

44. Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions, Dover Publications, 1970.

45. Nagel, C. M., Jr., "AIRYC - A Complex Airy Function Subroutine," Bell Telephone Laboratories, Technical Memorandum, MM 67-4113-2, Nov. 1967.

46. Stroud, A. H., Approximate Calculation of Multiple Integrals, Prentice-Hall, 1971.

APPENDICES

APPENDIX A

ENERGY EQUATION FOR LIQUID

The equation of state for a liquid treated as a pure substance is of the functional form

$$\bar{h}_1 = \bar{h}_1(\bar{p}_1, \bar{T}_1) \tag{A.1}$$

where the notation of Chapter II is preserved. A small change in the enthalpy $\bar{h}_1$ for a closed system can be expressed as

$$dh_1 = \left.\frac{\partial \bar{h}_1}{\partial \bar{p}_1}\right|_{\bar{T}_1} d\bar{p}_1 + \left.\frac{\partial \bar{h}_1}{\partial \bar{T}_1}\right|_{\bar{p}_1} d\bar{T}_1 \tag{A.2}$$

Modifying Eq. (A.2) for a differential control volume moving with a fluid particle the result (stated without proof) is

$$\frac{D\bar{h}_1}{Dt} = \left.\frac{\partial \bar{h}_1}{\partial \bar{p}_1}\right|_{\bar{T}_1} \frac{D\bar{p}_1}{Dt} + \left.\frac{\partial \bar{h}_1}{\partial \bar{T}_1}\right|_{\bar{p}_1} \frac{D\bar{T}_1}{Dt} \tag{A.3}$$

For a pure substance

$$\left.\frac{\partial \bar{h}_1}{\partial \bar{T}_1}\right|_{\bar{p}_1} = c_{p1} \tag{A.4}$$

and

$$\left.\frac{\partial \bar{h}_1}{\partial \bar{p}_1}\right|_{\bar{T}_1} = \frac{1}{\rho_1}(1 - \beta_1 \bar{T}_1) \tag{A.5}$$

where β_1 is the coefficient of volume expansion and c_{p1} is the specific heat at constant pressure. Substituting Eqs. (A.4) and (A.5) into Eq. (A.3), using the definitions of substantial derivatives and combining with Eq. (2.3.4)', the result is

$$\rho_1 c_{p1} \frac{D\bar{T}_1}{Dt} = \beta_1 \bar{T}_1 \frac{D\bar{p}_1}{Dt} + k_1 \nabla^2 \bar{T}_1 \tag{A.6}$$

where viscous dissipation is neglected. The discussion in Chapter I indicates that the pressure gradient in the present problem should be small. Also, the term $\beta_1 \bar{T}_1$ is small (e.g. for water $\beta_1 = 304 \times 10^{-6}$/deg C and for $\bar{T}_1$ in the range of 100 deg C, $\beta_1 \bar{T}_1 \simeq 0.03$) and therefore the first term on the right hand side of Eq. (A.6) can be neglected. This equation then reduces to

$$\rho_1 c_{p1} \frac{D\bar{T}_1}{Dt} = k_1 \nabla^2 \bar{T}_1 \tag{A.7}$$

which is a compact form of Eq. (2.3.4).

APPENDIX B

DERIVATION OF SHEAR AND NORMAL STRESS EQUATIONS

Consider equilibrium of a triangular element of fluid at the interface (Fig. B.1). Resolving all the forces normal and tangential to Δs and summing up, the following expressions are obtained:

$$\sigma = \sigma_{xx}\sin^2\phi + \sigma_{yy}\cos^2\phi - 2\tau_{xy}\sin\phi\cos\phi \tag{B.1}$$

$$\tau = \frac{1}{2}(\sigma_{yy} - \sigma_{xx})\sin 2\phi + \tau_{xy}(\cos^2\phi - \sin^2\phi) \tag{B.2}$$

For an incompressible fluid the stress tensor is given by

$$\begin{aligned} \sigma_{xx} &= -p + 2\mu\frac{\partial u}{\partial x} \\ \sigma_{yy} &= -p + 2\mu\frac{\partial v}{\partial y} \\ \tau_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{aligned} \tag{B.3}$$

From the geometry of Fig. B.1, $\tan\phi = \eta_x$, so that

$$\sin\phi = \frac{\eta_x}{(1 + \eta_x^2)^{1/2}}$$

and (B.4)

$$\cos\phi = \frac{1}{(1 + \eta_x^2)^{1/2}}$$

Introduction of Eqs. (B.3) and (B.4) into Eqs. (B.1) and (B.2) yields, after some simplifications,

$$\sigma = -p + \frac{2\mu}{(1 + \eta_x^2)} \left[\frac{\partial u}{\partial x} \eta_x^2 + \frac{\partial v}{\partial y} \right] - \frac{2\mu\eta_x}{(1 + \eta_x^2)} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] \quad (B.5)$$

and

$$\tau = \frac{1 - \eta_x^2}{1 + \eta_x^2} \mu \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] - \frac{2\mu\eta_x}{1 + \eta_x^2} \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] \quad (B.6)$$

These expressions are utilized in Chapters II and III.

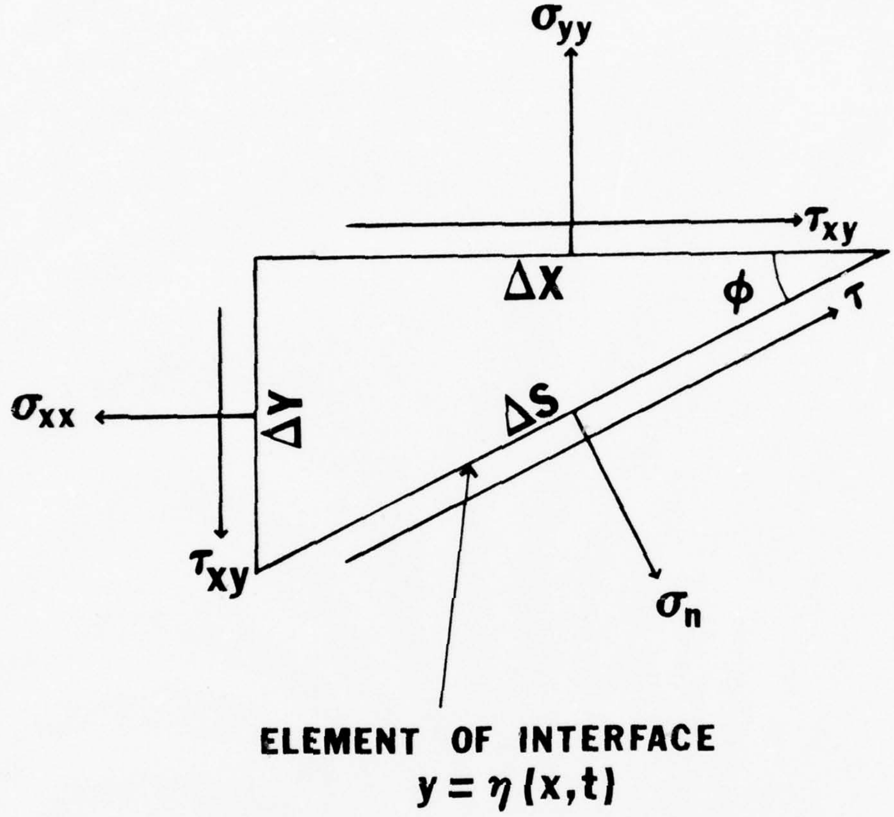

FIG. B.1 EQUILIBRIUM OF A FLUID ELEMENT AT THE INTERFACE

APPENDIX C

DERIVATIVES OF GENERAL SOLUTIONS WITHOUT MASS TRANSFER

Derivatives of $\psi_1(\xi)$ and $\psi_2(\eta)$ in Eqs. (4.3.17) and (4.3.19) w.r.t ξ and η, obtained using Leibniz's rule are

$$\psi_1'(\xi) = C_1\alpha_1 e^{\alpha_1\xi} - C_2\alpha_1 e^{-\alpha_1\xi} + C_3\int_{\xi^*}^{\xi}\cosh\{\alpha_1(\xi - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^+\}d\tilde{t}$$
$$+ C_4\int_{\xi^*}^{\xi}\cosh\{\alpha_1(\xi - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^-\}d\tilde{t} \tag{C.1}$$

$$\psi_1''(\xi) = C_1\alpha_1^2 e^{\alpha_1\xi} + C_2\alpha_1^2 e^{-\alpha_1\xi} + C_3\alpha_1\int_{\xi^*}^{\xi}\sinh\{\alpha_1(\xi - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^+\}d\tilde{t}$$
$$+ C_3 Ai\{\zeta_1(\xi)E^+\} + C_4\alpha_1\int_{\xi^*}^{\xi}\sinh\{\alpha_1(\xi - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^-\}d\tilde{t}$$
$$+ C_4 Ai\{\zeta_1(\xi)E^-\} \tag{C.2}$$

$$\psi_1'''(\xi) = C_1\alpha_1^3 e^{\alpha_1\xi} - C_2\alpha_1^3 e^{-\alpha_1\xi} + C_3\alpha_1^2\int_{\xi^*}^{\xi}\cosh\{\alpha_1(\xi - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^+\}d\tilde{t}$$
$$+ C_3 Ai'\{\zeta_1(\xi)E^+\}\zeta_1'E^+ + C_4\alpha_1^2\int_{\xi^*}^{\xi}\cosh\{\alpha_1(\xi - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^-\}d\tilde{t}$$
$$+ C_4 Ai'\{\zeta_1(\xi)E^-\}\zeta_1'E^- \tag{C.3}$$

Similarly

$$\dot{\psi}_2(\eta) = C_5\alpha_2 e^{\alpha_2\eta} - C_6\alpha_2 e^{-\alpha_2\eta} + C_7\int_{\eta^*}^{\eta}\cosh\{\alpha_2(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^+\}d\tilde{t}$$

$$+ C_8\int_{\eta^*}^{\eta}\cosh\{\alpha_2(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^-\}d\tilde{t} \qquad (C.4)$$

$$\ddot{\psi}_2(\eta) = C_5\alpha_2^2 e^{\alpha_2\eta} + C_6\alpha_2^2 e^{-\alpha_2\eta} + C_7\alpha_2\int_{\eta^*}^{\eta}\sinh\{\alpha_2(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^+\}d\tilde{t}$$

$$+ C_7 Ai\{\zeta_2(\eta)E^+\} + C_8\alpha_2\int_{\eta^*}^{\eta}\sinh\{\alpha_2(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^-\}d\tilde{t}$$

$$+ C_8 Ai\{\zeta_2(\eta)E^-\} \qquad (C.5)$$

$$\dddot{\psi}_2(\eta) = C_5\alpha_2^3 e^{\alpha_2\eta} - C_6\alpha_2^3 e^{-\alpha_2\eta} + C_7\alpha_2^2\int_{\eta^*}^{\eta}\cosh\{\alpha_2(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^+\}d\tilde{t}$$

$$+ C_7 Ai'\{\zeta_2(\eta)E^+\}\dot{\zeta}_2 E^+ + C_8\alpha_2^2\int_{\eta^*}^{\eta}\cosh\{\alpha_2(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^-\}d\tilde{t}$$

$$+ C_8 Ai'\{\zeta_2(\eta)E^-\}\dot{\zeta}_2 E^- \qquad (C.6)$$

where it follows from Eqs. (4.3.4) and (4.3.20) that

$$\zeta_1' = -i(\alpha_1 R_1 \hat{u}_1')^{1/3} \tag{C.7}$$

and

$$\dot{\zeta}_2 = -i(\alpha_2 R_2 \dot{\hat{u}}_2)^{1/3} \tag{C.8}$$

APPENDIX D

DERIVATIVES OF GENERAL SOLUTIONS WITH MASS TRANSFER

Differentiating Eqs. (5.4.15) and (5.5.16) w.r.t. ξ and Eqs. (5.4.17), (5.5.22) and (5.5.27) w.r.t. η using Leibniz's rule; and carrying out the necessary simplifications. the results are

$$\psi_1'(\xi) = C_1\alpha_1 e^{\alpha_1\xi} - C_2\alpha_1 e^{-\alpha_1\xi} + C_3\int_{\xi^*}^{\xi} e^{R_h\tilde{t}/2}\cosh\{\alpha_1(\xi - \tilde{t})\}\mathrm{Ai}\{\zeta_1(\tilde{t})E^+\}d\tilde{t}$$
$$+ C_4\int_{\xi^*}^{\xi} e^{R_h\tilde{t}/2}\cosh\{\alpha_1(\xi - \tilde{t})\}\mathrm{Ai}\{\zeta_1(\tilde{t})E^-\}d\tilde{t} \qquad \text{(D.1)}$$

$$\psi_1''(\xi) = C_1\alpha_1^2 e^{\alpha_1\xi} + C_2\alpha_1^2 e^{-\alpha_1\xi} + C_3\alpha_1\int_{\xi^*}^{\xi} e^{R_h\tilde{t}/2}\sinh\{\alpha_1(\xi - \tilde{t})\}\mathrm{Ai}\{\zeta_1(\tilde{t})E^+\}d\tilde{t}$$
$$+ C_3 e^{R_h\xi/2}\mathrm{Ai}\{\zeta_1(\xi)E^+\} + C_4\alpha_1\int_{\xi^*}^{\xi} e^{R_h\tilde{t}/2}\sinh\{\alpha_1(\xi - \tilde{t})\}\mathrm{Ai}\{\zeta_1(\tilde{t})E^+\}d\tilde{t}$$
$$+ C_4 e^{R_h\xi/2}\mathrm{Ai}\{\zeta_1(\xi)E^-\} \qquad \text{(D.2)}$$

$$\psi_1'''(\xi) = C_1\alpha_1^3 e^{\alpha_1\xi} - C_2\alpha_1^3 e^{-\alpha_1\xi} + C_3\alpha_2^2\int_{\xi^*}^{\xi} e^{R_h\tilde{t}/2}\cosh\{\alpha_1(\xi - \tilde{t})\}\mathrm{Ai}\{\zeta_1(\tilde{t})E^+\}d\tilde{t}$$
$$+ C_3 e^{R_h\xi/2}\left[\frac{R_h}{2}\mathrm{Ai}\{\zeta_1(\xi)E^+\} + \mathrm{Ai}'\{\zeta_1(\xi)E^+\}\zeta_1' E^+\right] +$$

$$+ C_4\alpha_1^2\int_{\xi^*}^{\xi} e^{R_h\tilde{t}/2}\cosh\{\alpha_1(\xi - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^-\}d\tilde{t}$$

$$+ C_4 e^{R_h\xi/2}\left[\frac{R_h}{2}Ai\{\zeta_1(\xi)E^-\} + Ai'\{\zeta_1(\xi)E^-\}\zeta_1'E^-\right] \tag{D.3}$$

Similarly

$$\dot{\psi}_2(\eta) = C_5\alpha_2 e^{\alpha_2\eta} - C_6\alpha_2 e^{-\alpha_2\eta} + C_7\int_{\eta^*}^{\eta} e^{R_\delta\tilde{t}/2}\sinh\{\alpha_2(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^+\}d\tilde{t}$$

$$+ C_8\int_{\eta^*}^{\eta} e^{R_\delta\tilde{t}/2}\sinh\{\alpha_2(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^-\}d\tilde{t} \tag{D.4}$$

$$\ddot{\psi}_2(\eta) = C_5\alpha_2^2 e^{\alpha_2\eta} + C_6\alpha_2^2 e^{-\alpha_2\eta} + C_7\alpha_2\int_{\eta^*}^{\eta} e^{R_\delta\tilde{t}/2}\sinh\{\alpha_1(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^+\}d\tilde{t}$$

$$+ C_7 e^{R_\delta\eta/2}Ai\{\zeta_2(\eta)E^+\} + C_8\alpha_2\int_{\eta^*}^{\eta} e^{R_\delta\tilde{t}/2}\sinh\{\alpha_2(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^-\}d\tilde{t}$$

$$+ C_8 e^{R_\delta\eta/2}Ai\{\zeta_2(\eta)E^-\} \tag{D.5}$$

$$\dddot{\psi}_2(\eta) = C_5\alpha_2^3 e^{\alpha_2\eta} - C_6\alpha_2^3 e^{-\alpha_2\eta} + C_7\alpha_2^2\int_{\eta^*}^{\eta} e^{R_\delta\tilde{t}/2}\cosh\{\alpha_2(\eta - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^+\}d\tilde{t}$$

$$+ C_7 e^{R_\delta\eta/2}\left[\frac{R_\delta}{2}Ai\{\zeta_2(\eta)E^+\} + Ai'\{\zeta_2(\eta)E^+\}\dot{\zeta}_2E^+\right] +$$

$$+ C_8\alpha_2^2\int_{\eta^*}^{\eta} e^{R_\delta\tilde{t}/2}\cosh\{\alpha_2(\eta-\tilde{t})\}Ai\{\zeta_2(\tilde{t})E^-\}d\tilde{t}$$

$$+ C_8 e^{R_\delta\eta/2}\left[\frac{R_\delta}{2}Ai\{\zeta_2(\eta)E^-\} + Ai'\{\zeta_2(\eta)E^-\}\dot{\zeta}_2E^-\right] \quad (D.6)$$

$$\theta_1'(\xi) = e^{R_hPr_1\xi/2}\left[z_1'\left\{C_9Ai'\{z_1(\xi)E^+\}E^+ + C_{10}Ai'\ z_1(\xi)E^-\}E^-\right\}\right.$$

$$+\frac{2\pi Pr_1R_1\hat{T}_1'z_1'}{(\alpha_1Pr_1R_1\hat{u}_1')^{1/3}}\left.\left\{C_1J_1'(\xi) + C_2J_2'(\xi) + C_3J_3'(\xi) + C_4J_4'(\xi)\right\}\right]$$

$$+ e^{R_hPr_1\xi/2}\left[\frac{R_hPr_1}{2}\left\{C_9Ai\{z_1(\xi)E^+\} + C_{10}Ai\{z_1(\xi)E^-\}\right\}\right.$$

$$+\frac{2\pi Pr_1R_1\hat{T}_1'z_1'}{(\alpha_1Pr_1R_1\hat{u}_1')^{1/3}}\left.\left\{C_1J_1(\xi) + C_2J_2(\xi) + C_3J_3(\xi) + C_4J_4(\xi)\right\}\right]$$

(D.7)

where

$$J_{1,2}'(\xi) = \int_{\xi^*}^{\xi} e^{(\pm\alpha_1 - \frac{R_hPr_1}{2})\tilde{t}}\, G_1(\tilde{t};\xi)d\tilde{t} \quad (D.8)$$

$$J_{3,4}'(\xi) = \int_{\xi^*}^{\xi} e^{-R_hPr_1\tilde{t}/2}\, G_1(\tilde{t};\xi)\int_{\tilde{t}^*}^{\tilde{t}} e^{R_h\tilde{\tau}/2}\sinh\{\alpha_1(\tilde{t}-\tilde{\tau})\}Ai\{\zeta_1(\tilde{\tau})E^\pm\}d\tilde{\tau}d\tilde{t}$$

(D.9)

with

$$G_1(\tilde{t};\xi) = Ai\{z_1(\tilde{t})E^+\}Ai'\{z_1(\xi)E^-\}E^-$$

$$-Ai\{z_1(\tilde{t})E^-\}Ai'\{z_1(\xi)E^+\}E^+ \quad (D.10)$$

and

$$z_1' = -i(\alpha_1 Pr_1 R_1 \hat{u}_1')^{2/3} \tag{D.11}$$

Similarly

$$\dot{\theta}_2 = e^{R_\delta \eta/2}\left[\dot{z}_2\left\{ C_{11}Ai'\{z_2(\eta)E^+\}E^+ + C_{12}Ai'\{z_2(\eta)E^-\}E^-\right\}\right.$$

$$\left. + \frac{2\pi R_2 \dot{\hat{T}}_2 \dot{z}_2}{(\alpha_2 R_2 \dot{\hat{u}}_2)^{1/3}}\left\{C_5 \dot{J}_5(\eta) + C_6 \dot{J}_6(\eta) + C_7 \dot{J}_7(\eta) + C_8 \dot{J}_8(\eta)\right\}\right]$$

$$+ e^{R_\delta \eta/2}\left[\frac{R_\delta}{2}\left\{C_{11}Ai\{z_2(\eta)E^+\} + C_{12}Ai\{z_2(\eta)E^-\}\right\}\right.$$

$$\left. + \frac{2\pi R_2 \dot{\hat{T}}_2}{(\alpha_2 R_2 \dot{\hat{u}}_2)^{1/3}}\left\{C_5 J_5(\eta) + C_6 J_6(\eta) + C_7 J_7(\eta) + C_8 J_8(\eta)\right\}\right] \tag{D.12}$$

where

$$\dot{J}_{5,6}(\eta) = \int_{\eta^*}^{\eta} e^{(\pm\alpha_2 - \frac{R_\delta}{2})\tilde{t}}\ G_2(\tilde{t};\eta)d\tilde{t} \tag{D.13}$$

$$\dot{J}_{7,8}(\eta) = \int_{\eta^*}^{\eta} e^{-R_\delta \tilde{t}/2}\ G_2(\tilde{t};\eta)\int_{\tilde{t}^*}^{\tilde{t}} e^{R_\delta \tilde{\tau}/2}\ \sinh\{\alpha_2(\tilde{t} - \tilde{\tau})\}Ai\{\zeta_2(\tilde{\tau})E^{\pm}\}d\tilde{\tau}d\tilde{t} \tag{D.14}$$

with

$$G_2(\tilde{t};\xi) = Ai\{z_2(\tilde{t})E^+\}Ai'\{z_2(\eta)E^-\}E^-$$

$$-Ai\{z_2(\tilde{t})E^-\}Ai'\{z_2(\xi)E^+\}E^+ \qquad (D.15)$$

and

$$\dot{z}_2 = -i(\alpha_2 R_2 \dot{\hat{u}}_2)^{2/3} \qquad (D.16)$$

Finally,

$$\dot{\chi}(\eta) = e^{R_\delta \eta/2}\left[\dot{z}_2\left\{C_{13}Ai'\{z_2(\eta)E^+\}E^+ + C_{14}Ai'\{z_2(\eta)E^-\}E^-\right\}\right.$$

$$\left. + \frac{2\pi R_2 \dot{\hat{\chi}}\dot{z}_2}{(\alpha_2 R_2 \dot{\hat{u}}_2)^{1/3}}\left\{C_5\dot{J}_5(\eta) + C_6\dot{J}_6(\eta) + C_7\dot{J}_7(\eta) + C_8\dot{J}_8(\eta)\right\}\right]$$

$$+ e^{R_\delta \eta/2}\left[\frac{R_\delta}{2}\left\{C_{13}Ai\{z_2(\eta)E^+\} + C_{14}Ai\{z_2(\eta)E^-\}\right\}\right.$$

$$\left. + \frac{2\pi R_2 \dot{\hat{\chi}}}{(\alpha_2 R_2 \dot{\hat{u}}_2)^{1/3}}\left\{C_5 J_5(\eta) + C_6 J_6(\eta) + C_7 J_7(\eta) + C_8 J_8(\eta)\right\}\right] \qquad (D.17)$$

with Eqs. (D.13) - (D.15) applying in this case also.

APPENDIX E

LIST OF INTEGRALS IN ZERO MASS TRANSFER PROBLEM

The integrals I_1 through I_{16} in Eqs. (4.4.1) through (4.4.17) are defined as follows:

$$I_{1,2} = \int_{\xi^*}^{-1} \sinh\{\alpha_1(-1 - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{3,4} = \int_{\xi^*}^{-1} \cosh\{\alpha_1(-1 - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{5,6} = \int_{\eta^*}^{1} \sinh\{\alpha_2(1 - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{7,8} = \int_{\eta^*}^{1} \cosh\{\alpha_2(1 - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{9,10} = \int_{\xi^*}^{0} \sinh(\alpha_1\tilde{t})Ai\{\zeta_1(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{11.12} = \int_{\xi^*}^{0} \cosh(\alpha_1\tilde{t})Ai\{\zeta_1(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{13,14} = \int_{\eta^*}^{0} \sinh(\alpha_2\tilde{t})Ai\{\zeta_2(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{15,16} = \int_{\eta^*}^{0} \cosh(\alpha_2\tilde{t})Ai\{\zeta_2(\tilde{t})E^{\pm}\}d\tilde{t}$$

with ξ^* and η^* such that $\zeta_1(\xi^*) = 0$ and $\zeta_2(\eta^*) = 0$.

APPENDIX F

LIST OF INTEGRALS IN MASS TRANSFER PROBLEM

The integrals I_1 through I_{48} in Eqs. (5.6.1) through (5.6.15) are defined as follows:

$$I_{1,2} = \int_{\xi^*}^{-1} e^{R_h\tilde{t}/2} \sinh\{\alpha_1(-1 - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{3,4} = \int_{\xi^*}^{-1} e^{R_h\tilde{t}/2} \cosh\{\alpha_1(-1 - \tilde{t})\}Ai\{\zeta_1(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{5,6} = \int_{\eta^*}^{1} e^{R_\delta\tilde{t}/2} \sinh\{\alpha_2(1 - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{7,8} = \int_{\eta^*}^{1} e^{R_\delta\tilde{t}/2} \cosh\{\alpha_2(1 - \tilde{t})\}Ai\{\zeta_2(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{9,10} = \int_{\xi^*}^{0} e^{R_h\tilde{t}/2} \sinh(\alpha_1\tilde{t})Ai\{\zeta_1(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{11,12} = \int_{\xi^*}^{0} e^{R_h\tilde{t}/2} \cosh(\alpha_1\tilde{t})Ai\{\zeta_1(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{13,14} = \int_{\eta^*}^{0} e^{R_\delta\tilde{t}/2} \sinh(\alpha_2\tilde{t})Ai\{\zeta_2(\tilde{t})E^{\pm}\}d\tilde{t}$$

$$I_{15,16} = \int_{\eta^*}^{0} e^{R_\delta\tilde{t}/2} \cosh(\alpha_2\tilde{t})Ai\{\zeta_2(\tilde{t})E^{\pm}\}d\tilde{t}$$

In integrals I_{1-16}, ξ^* and η^* are such that $\zeta_1(\xi^*) = 0$ and $\zeta_2(\eta^*) = 0$.

$$I_{17,18} = J_{1,2}(-1) = \int_{\xi^*}^{-1} e^{(\pm\alpha_1 - \frac{R_h Pr_1}{2})\tilde{t}} F_1(\tilde{t};-1)d\tilde{t}$$

$$I_{19,20} = J'_{1,2}(-1) = \int_{\xi^*}^{-1} e^{(\pm\alpha_1 - \frac{R_h Pr_1}{2})\tilde{t}} G_1(\tilde{t};-1)d\tilde{t}$$

$$I_{21,22} = J_{5,6}(1) = \int_{\eta^*}^{1} e^{(\pm\alpha_2 - \frac{R_\delta}{2})\tilde{t}} F_2(\tilde{t};1)d\tilde{t}$$

$$I_{23,24} = \dot{J}_{5,6}(1) = \int_{\eta^*}^{1} e^{(\pm\alpha_2 - \frac{R_\delta}{2})\tilde{t}} G_2(\tilde{t};1)d\tilde{t}$$

$$I_{25,26} = J_{1,2}(0) = \int_{\xi^*}^{0} e^{(\pm\alpha_1 - \frac{R_h Pr_1}{2})\tilde{t}} F_1(\tilde{t};0)d\tilde{t}$$

$$I_{27,28} = J'_{1,2}(0) = \int_{\xi^*}^{0} e^{(\pm\alpha_1 - \frac{R_h Pr_1}{2})\tilde{t}} G_1(\tilde{t};0)d\tilde{t}$$

$$I_{29,30} = J_{5,6}(0) = \int_{\eta^*}^{0} e^{(\pm\alpha_2 - \frac{R_\delta}{2})\tilde{t}} F_2(\tilde{t};0)d\tilde{t}$$

$$I_{31,32} = \dot{J}_{5,6}(0) = \int_{\eta^*}^{0} e^{(\pm\alpha_2 - \frac{R_\delta}{2})\tilde{t}} G_2(\tilde{t};0)d\tilde{t}$$

In integrals I_{17-32}, ξ^* and η^* are such that $z_1(\xi^*) = 0$ and $z_2(\eta^*) = 0$. F_1, F_2, G_1 and G_2 are given by Eqs. (5.5.19), (5.5.25), (D.10) and (D.15) respectively.

And finally,

$$I_{33,34} = J_{3,4}(-1) = \int_{\eta^*}^{-1} e^{-R_h Pr_1 \tilde{t}/2} F_1(\tilde{t};-1)d_1(\tilde{t})d\tilde{t}$$

$$I_{35,36} = J'_{3,4}(-1) = \int_{\eta^*}^{-1} e^{-R_h Pr_1 \tilde{t}/2} G_1(\tilde{t};-1)d_1(\tilde{t})d\tilde{t}$$

$$I_{37,38} = J_{7,8}(1) = \int_{\eta^*}^{1} e^{-R_\delta \tilde{t}/2}\, F_2(\tilde{t};-1)\, d_2(\tilde{t})\, d\tilde{t}$$

$$I_{39,40} = \dot{J}_{7,8}(1) = \int_{\eta^*}^{1} e^{-R_\delta \tilde{t}/2}\, G_2(\tilde{t};-1)\, d_2(\tilde{t})\, d\tilde{t}$$

$$I_{41,42} = J_{3,4}(0) = \int_{\xi^*}^{0} e^{-R_h Pr_1 \tilde{t}/2}\, F_1(\tilde{t};0)\, d_1(\tilde{t})\, d\tilde{t}$$

$$I_{43,44} = J'_{3,4}(0) = \int_{\xi^*}^{0} e^{-R_h Pr_1 \tilde{t}/2}\, G_1(\tilde{t};0)\, d_1(\tilde{t})\, d\tilde{t}$$

$$I_{45,46} = J_{7,8}(0) = \int_{\eta^*}^{0} e^{-R_\delta \tilde{t}/2}\, F_2(\tilde{t};0)\, d_2(\tilde{t})\, d\tilde{t}$$

$$I_{47,48} = \dot{J}_{7,8}(0) = \int_{\eta^*}^{0} e^{-R_\delta \tilde{t}/2}\, G_2(\tilde{t};0)\, d_2(\tilde{t})\, d\tilde{t}$$

where

$$d_1(\tilde{t}) = \int_{\tilde{t}^*}^{\tilde{t}} e^{R_h \tilde{\tau}/2}\, \sinh\{\alpha_1(\tilde{t} - \tilde{\tau})\}\, Ai\{\zeta_1(\tilde{\tau})E^{\pm}\}\, d\tilde{\tau} \tag{F.1}$$

and

$$d_2(\tilde{t}) = \int_{\tilde{t}^*}^{\tilde{t}} e^{R_\delta \tilde{\tau}/2}\, \sinh\{\alpha_2(\tilde{t} - \tilde{\tau})\}\, Ai\{\zeta_2(\tilde{\tau})E^{\pm}\}\, d\tilde{\tau} \tag{F.2}$$

For integrals I_{33-48}, ξ^*, η^* and $\tilde{t}^*$ are defined such that $z_1(\xi^*) = \zeta_1(t^*) = 0$ and $\zeta_2(t^*) = z_2(t^*) = 0$. In the integrals I_1 through I_{48}; ζ_1, ζ_2, z_1 and z_2 are the transformations given by Eqs. (5.4.8), (5.4.18), (5.5.3) and (5.4.18) respectively. These notations used in the mass transfer problem are the same as in Chapter IV in order to facilitate comparisons.

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HYDRODYNAMIC STABILITY OF LIQUID FILMS ADJACENT TO INCOMPRESSIBLE GAS STREAMS INCLULING, ETC...(U) PRAKASH B. JOSHI, ET AL.

UNCLASSIFIED VIRGINIA POLYTECHNIC INST. & STATE UNIV., BLACKSBURG. COLL. OF ENG.

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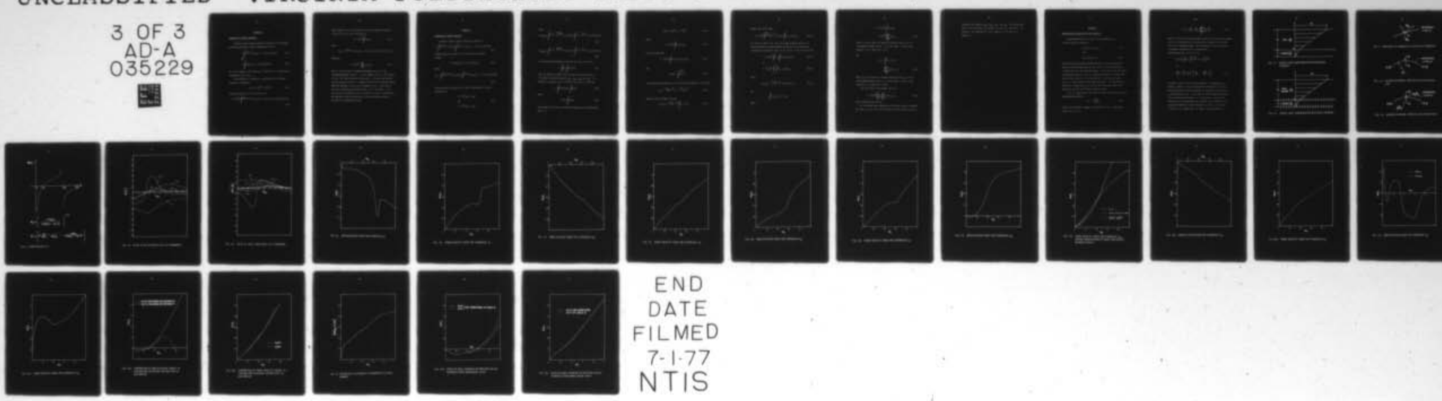

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APPENDIX G

EVALUATION OF SINGLE INTEGRALS

Consider typical single integrals encountered in the problems with and without mass transfer (Appendices E and F)

$$I = \int_{\xi^*}^{\xi} e^{R_h \tilde{t}/2} \sinh\{\alpha_1(\xi - \tilde{t})\} Ai\{\zeta_1(\tilde{t})E^{\pm}\} d\tilde{t} \tag{G.1}$$

and

$$I = \int_{\xi^*}^{\xi} \sinh\{\alpha_1(\xi - \tilde{t})\} Ai\{\zeta_1(\tilde{t})E^{\pm}\} d\tilde{t} \tag{G.2}$$

Eq. (G.1) reduces to (G.2) when $R_h = 0$ and hence it is sufficient to concentrate on Eq. (G.1).

A new variable of integration t_1 defined by the following equation is introduced:

$$\tilde{t}(t_1) = \frac{\xi + \xi^*}{2} + \frac{\xi - \xi^*}{2} t_1 \tag{G.3}$$

Then the integral (G.1) can be written as

$$I = \frac{\xi - \xi^*}{2} \int_{-1}^{1} e^{R_h \tilde{t}(t_1)/2} \sinh\,[\alpha_1\{\xi - \tilde{t}(t_1)\}]\; Ai\,[\,\zeta_1\{\tilde{t}(t_1)\}E^{\pm}\,]\, dt_1 \tag{G.4}$$

This integral is in a form suitable for Gauss-Legendre integration. In fact, Eq. (G.4) can be expressed as

$$I = \frac{\xi - \xi^*}{2} \int_{-1}^{1} f(t_1) dt_1 \tag{G.5}$$

where

$$f(t_1) = e^{R_h \tilde{t}(t_1)} \sinh [\alpha_1 \{\xi - \tilde{t}(t_1)\}] \; Ai \; [\zeta_1 \{\tilde{t}(t_1)\} E^{\pm}] \tag{G.6}$$

Therefore,

$$I = \frac{\xi - \xi^*}{2} \sum_{i=1}^{n} w_i f(\gamma_i) \tag{G.7}$$

where γ_i are the zeros of Legendre polynomials and w_i are the corresponding weight factors. n is the number of zeros. The function f in Eq. (G.6) was calculated at the zeros γ_i using an Airy function routine described in Re. 45. A modified version of the IBM SSP-routine DQG32 was employed to carry out the summation (G.7). Solution for the final eigenvalue was obtained using n = 16, 32 and 96. It was observed that the difference in the values of c_1 with 32 and 96 points was negligibly small and therefore a 32-point scheme was adopted throughout the computational work.

APPENDIX H

EVALUATION OF DOUBLE INTEGRALS

Consider a typical double integral in Appendix F:

$$I = \int_{\xi^*}^{\xi} e^{-R_h Pr_1 \tilde{t}/2} F_1(\tilde{t};\xi) \int_{\tilde{t}^*}^{\tilde{t}} e^{R_h \tilde{\tau}/2} \sinh\{\alpha_1(\tilde{t} - \tilde{\tau})\} Ai\{\zeta_1(\tilde{\tau})E^+\} d\tilde{\tau} d\tilde{t} \tag{H.1}$$

Substituting for F_1 from Eq. (5.5.19) the above integral becomes

$$I = Ai\{z_1(\xi)E^-\} I_1 - Ai\{z_1(\xi)E^+\} I_2 \tag{H.2}$$

where

$$I_{1,2} = \int_{\xi^*}^{\xi} e^{-R_h Pr_1 \tilde{t}/2} Ai\{z_1(\tilde{t})E^{\pm}\} \int_{\tilde{t}^*}^{\tilde{t}} e^{R_h \tilde{\tau}/2} \sinh\{\alpha_1(\tilde{t} - \tilde{\tau})\} Ai\{\zeta_1(\tilde{\tau})E^+\} d\tilde{\tau} d\tilde{t} \tag{H.3}$$

By writing the sinh function in terms of exponentials it may be verified that

$$I_1 = \frac{1}{2} (S_{1AA} - S_{2AA})$$

and (H.4)

$$I_2 = \frac{1}{2} (S_{1BA} - S_{2BA})$$

where

$$S_{1AA,2AA} = \int_{\xi^*}^{\xi} e^{(\pm\alpha_1 - \frac{R_h Pr_1}{2})\tilde{t}} \, Ai\{z_1(\tilde{t})E^+\} \int_{\tilde{t}^*}^{\tilde{t}} e^{(\pm\alpha + \frac{R_h}{2})\tilde{\tau}} \, Ai\{\zeta_1(\tilde{\tau})E^+\} d\tilde{\tau} d\tilde{t} \tag{H.5}$$

$$S_{1BA,2BA} = \int_{\xi^*}^{\xi} e^{(\pm\alpha_1 - \frac{R_h Pr_1}{2})\tilde{t}} \, Ai\{z_1(\tilde{t})E^-\} \int_{\tilde{t}^*}^{\tilde{t}} e^{(\pm\alpha + \frac{R_h}{2})\tilde{\tau}} \, Ai\{\zeta_1(\tilde{\tau})E^+\} d\tilde{\tau} d\tilde{t} \tag{H.6}$$

It is noticed that integrals (H.5) and (H.6) are of the form

$$S = \int_{\xi^*}^{\xi} A(\tilde{t}) \int_{\tilde{t}^*}^{\tilde{t}} B(\tilde{\tau}) d\tilde{\tau} d\tilde{t} \tag{H.7}$$

Thus the original integral (H.1) has been expressed in terms of four simpler iterated integrals S_{1AA}, S_{2AA}, S_{1BA} and S_{2BA}. It is possible to simplify (H.7) further by rewriting it in the form

$$S = \int_{\xi^*}^{\xi} A(\tilde{t}) C(\tilde{t}) d\tilde{t} = \int_{\xi^*}^{\xi} D(\tilde{t}) d\tilde{t} \tag{H.8}$$

where

$$C(\tilde{t}) = \int_{\tilde{t}^*}^{\tilde{t}} B(\tilde{\tau}) d\tilde{\tau} \tag{H.9}$$

The integral (H.8) can be evaluated by employing the transformation (G.3), i.e.

$$\tilde{t}(t_1) = \frac{\xi - \xi^*}{2} t_1 + \frac{\xi + \xi^*}{2} \tag{H.10}$$

Hence

$$S = \frac{\xi - \xi^*}{2} \int_{-1}^{1} D\{\tilde{t}(t_1)\} dt_1 \tag{H.11}$$

It then follows that

$$S = \frac{\xi - \xi^*}{2} \int_{-1}^{1} A\{\tilde{t}(t_1)\} C\{\tilde{t}(t_1)\} dt_1 \tag{H.12}$$

Thus

$$C\{\tilde{t}(t_1)\} = \int_{\tilde{t}^*}^{\tilde{t}(t_1)} B(\tilde{\tau}) d\tilde{\tau} \tag{H.13}$$

The following transformation is now introduced analogous to Eq. (H.10),

$$\tilde{\tau}(\tau_1) = \frac{\tilde{t}(t_1) - \tilde{t}^*}{2} \tau_1 + \frac{\tilde{t}(\tilde{t}_1) + \tilde{t}^*}{2} \tag{H.14}$$

Therefore (H.13) assumes the form

$$C\{\tilde{t}(t_1)\} = \frac{\tilde{t}(t_1) - \tilde{t}^*}{2} \int_{-1}^{1} B\{\tilde{\tau}(\tau_1)\} d\tau_1 \tag{H.15}$$

Finally, Eq. (H.7) reads

$$S = \frac{\xi - \xi^*}{2} \int_{-1}^{1} \frac{\tilde{t}(t_1) - \tilde{t}^*}{2} A\{\tilde{t}(t_1)\} \int_{-1}^{1} B\{\tilde{\tau}(\tau_1)\} d\tau_1 dt_1 \tag{H.16}$$

Eq. (H.16) is akin to Eq. (G.4) for single integrals and is in a form convenient for Gauss-Legendre integration in two dimensions. Following the procedure in Ref. 46, Eq. (H.16) can be put in the form

$$S = \frac{\xi - \xi^*}{2} \int_{-1}^{1} f(t_1) \int_{-1}^{1} g(t_1, \tau_1) d\tau_1 dt_1 \tag{H.17}$$

or

$$S = \frac{\xi - \xi^*}{2} \int_{-1}^{1} \int_{-1}^{1} h(t_1, \tau_1) d\tau_1 dt_1 \tag{H.18}$$

where

$$h(t_1, \tau_1) = \frac{\tilde{t}(t_1) - \tilde{t}^*}{2} A\{\tilde{t}(t_1)\} B\{\tilde{\tau}(\tau_1)\} \tag{H.19}$$

Let

$$\int_{-1}^{1} h(t_1, \tau_1) d\tau_1 = \kappa(t_1)$$

Hence

$$S = \frac{\xi - \xi^*}{2} \int_{-1}^{1} \kappa(t_1)\,dt_1$$

$$= \frac{\xi - \xi^*}{2} \sum_{i=1}^{n_1} w_i \kappa(\gamma_i) \tag{H.21}$$

where γ_i are the zeros of Legendre polynomials and w_i are the corresponding weight factors. n_1 is the number of zeros in the interval (-1, 1) along the t_1 axis.

Now

$$\kappa(\gamma_i) = \int_{-1}^{1} h(\gamma_i, \tau_1)\,d\tau_1$$

$$= \sum_{j=1}^{n_2} w_{ij} h(\gamma_i, \bar{\gamma}_{ij}) \tag{H.22}$$

where $\bar{\gamma}_{ij}$ are the zeros of Legendre polynomials and w_{ij} are the corresponding weight factors. n_2 is the number of zeros in the interval (-1, 1) along the τ_1 axis.

The final form of the integral (H.16) is

$$S = \frac{\xi - \xi^*}{2} \sum_{i=1}^{n_1} w_i \sum_{j=1}^{n_2} w_{ij} h(\gamma_i, \bar{\gamma}_{ij}) \tag{H.23}$$

with h defined by Eq. (H.19).

In the present work, experience with single integrals suggested the choice $n_1 = n_2 = 32$. The procedure described above was used to calculate the integrals S_{1AA}, S_{2AA}, S_{1BA}, and S_{2BA}. This computation leads to the evaluation of I through equations (H.4) and (H.2). The details of the computation of the integrand are the same as in Appendix G.

APPENDIX I

NEWTON-RAPHSON ITERATION FOR THE EIGENVALUE

As mentioned in Secs. 4.5 and 5.7 the eigenvalue c_1 is evaluated using the equations,

$$[A(c_1)]\{C\} = \{V(c_1)\} \tag{I.1}$$

and

$$G[c_1,\{C(c_1)\}] = 0 \tag{I.2}$$

For the zero mass transfer problem, $[A(c_1)]$ is an 8 x 8 coefficient matrix of the left hand sides of Eqs. (4.4.1) - (4.4.8) and $V(c_1)$ is a column matrix of the right hand sides. In the case of the mass transfer problem, $[A(c_1)]$ is a 14 x 14 coefficient matrix of the left hand sides of Eqs. (5.6.1) - (5.6.7) and (5.6.9) - (5.6.15), and again $V(c_1)$ is a column matrix of the right hand sides. The characteristic function G is given by Eq. (4.4.11) for the zero mass transfer case and by Eq. (5.6.17) for the mass transfer problem.

If c_1 is a guess value, then the first order correction due to the Newton-Raphson method is

$$\Delta c_1 = -\frac{G(c_1)}{G'(c_1)} \tag{I.3}$$

Thus it is necessary to compute the derivative $G'(c_1)$. Differentiating (I.2) w.r.t. c_1,

$$G'(c_1) = \frac{dG}{dc_1} = \frac{\partial G}{\partial c_1} + \sum_{i=1}^{N} \frac{\partial G}{\partial C_i} \frac{\partial C_i}{\partial c_1} \qquad (I.4)$$

where N = 8 for the zero mass transfer and N = 14 for the mass transfer case. The calculation of $\partial G/\partial c_1$ and $\partial G/\partial C_i$ from Eqs. (4.4.11) and (5.6.17) is straight-forward. The calculation of $\partial C_i/\partial c_1$, however, is somewhat involved and it is outlined here.

Differentiating Eq. (I.1) w.r.t. c_1,

$$\left[A(c_1)\right] \frac{\partial\{C\}}{\partial c_1} + \frac{\partial\left[A(c_1)\right]}{\partial c_1}\{C\} = \frac{\partial\{V(c_1)\}}{\partial c_1}$$

or

$$\frac{\partial\{C\}}{\partial c_1} = \frac{\partial C_i}{\partial c_1} = \left[A\right]^{-1}\left[\frac{\partial V}{\partial c_1} - \frac{\partial[A]}{\partial c_1}\{C\}\right] \qquad (I.5)$$

As before, $\{\partial V/\partial c_1\}$ can be easily obtained. The computation of $\partial[A]/\partial c_1$, however, is a very complicated task because it involves differentiating the various integrals (in Appendices E and F) w.r.t. c_1 using Leibniz's rule. This leads to tedious algebra and therefore the details are omitted. It is sufficient to say that the derivatives of the above-mentioned integrals can be related to the integrals themselves through integration by parts. Once $\partial[A]/\partial c_1$ is known it is a simple matter to compute $G'(c_1)$ and hence Δc_1.

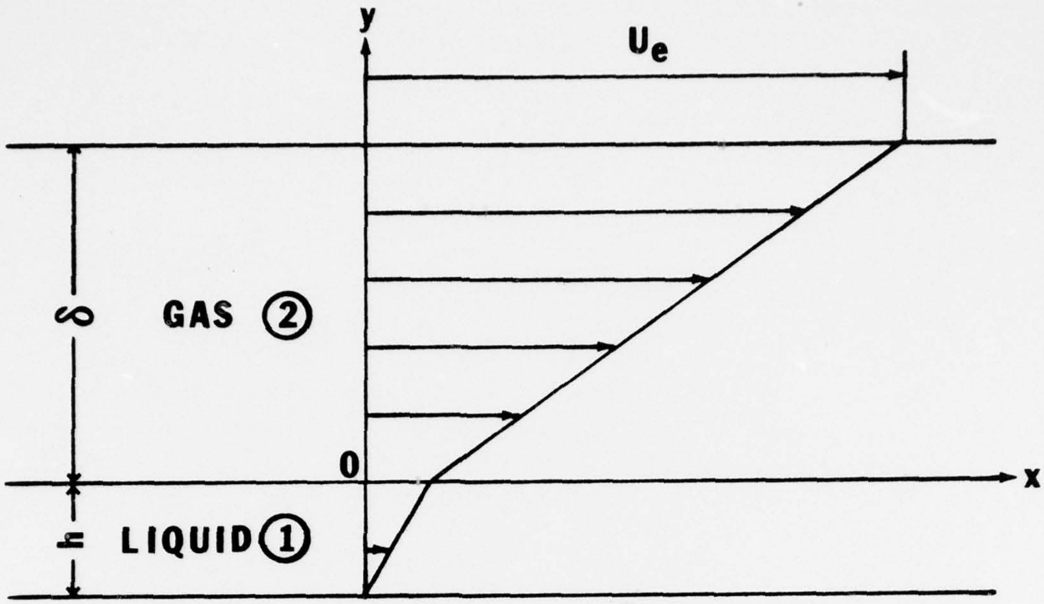


FIG. 1a. STEADY-STATE CONFIGURATION WITHOUT MASS TRANSFER

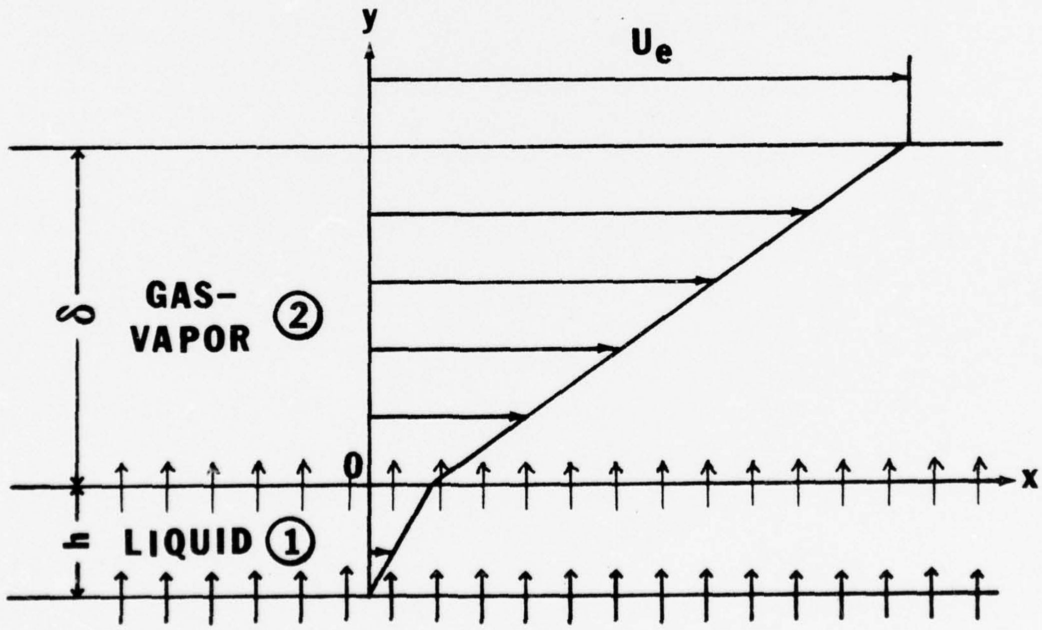


FIG. 1b. STEADY-STATE CONFIGURATION WITH MASS TRANSFER

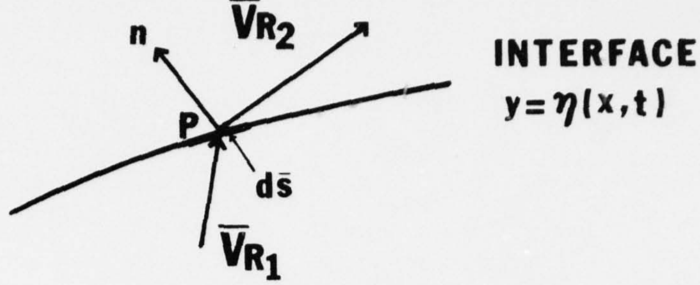


FIG. 2 CONTINUITY OF TANGENTIAL VELOCITIES AT INTERFACE

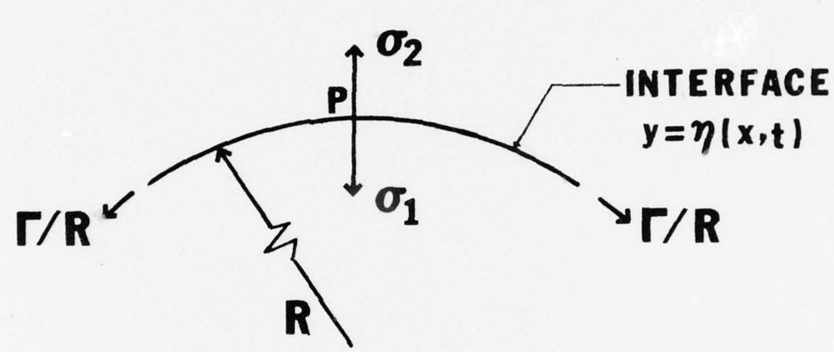


FIG. 3 a. BALANCE OF NORMAL STRESSES (zero mass transfer case)

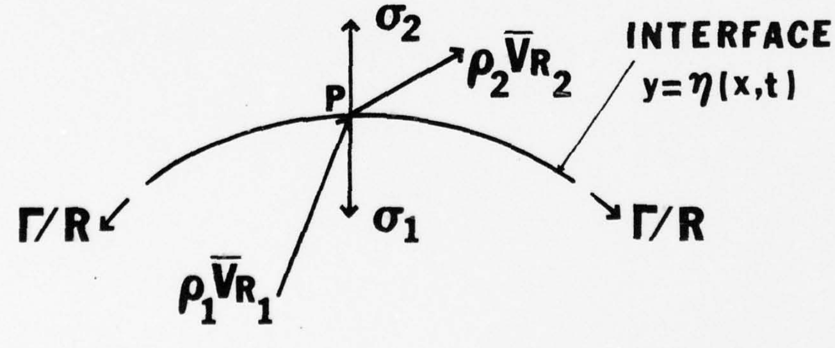
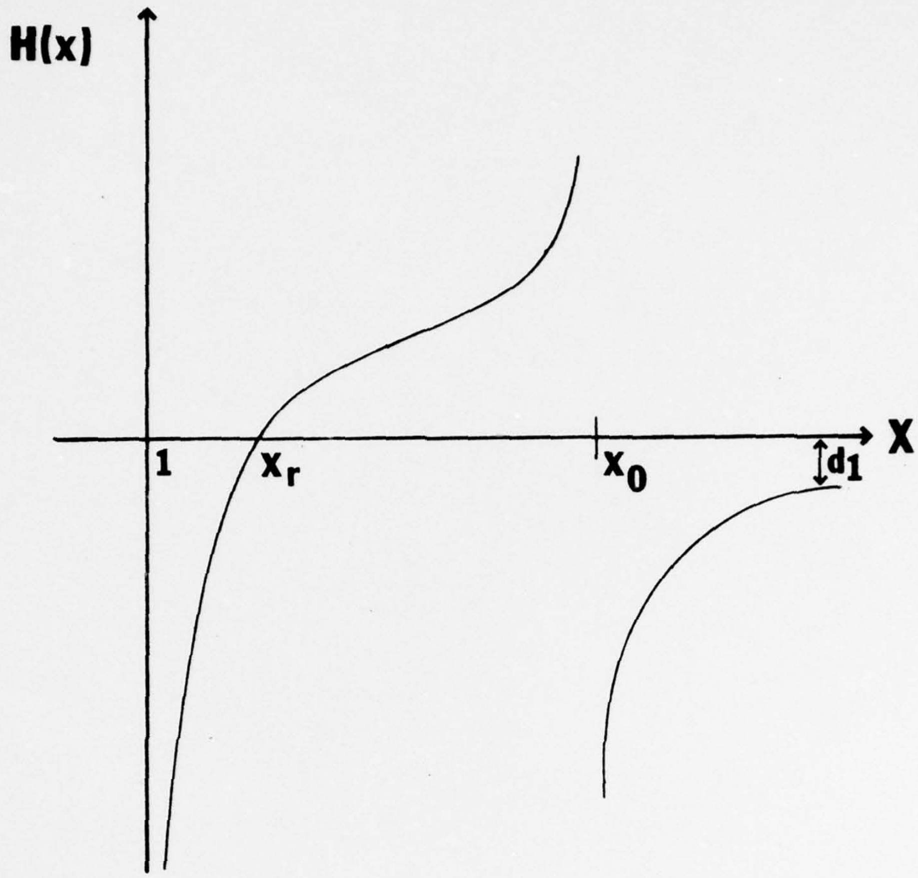


FIG. 3 b. BALANCE OF NORMAL STRESSES (mass transfer case)



$$X_0 = \left[\frac{1/C_{p1}T_e}{1/C_{p1}T_e - T_w/T_e} \right]^{1/n}$$

$$d_1 = \left[\ln \frac{P_e}{P_{ref}} - \frac{1}{RT_{ref}} - \frac{1/RT_e}{1/C_{p1}T_e - T_w/T_e} \right]$$

FIG. 4 FUNCTION H(x) VS x

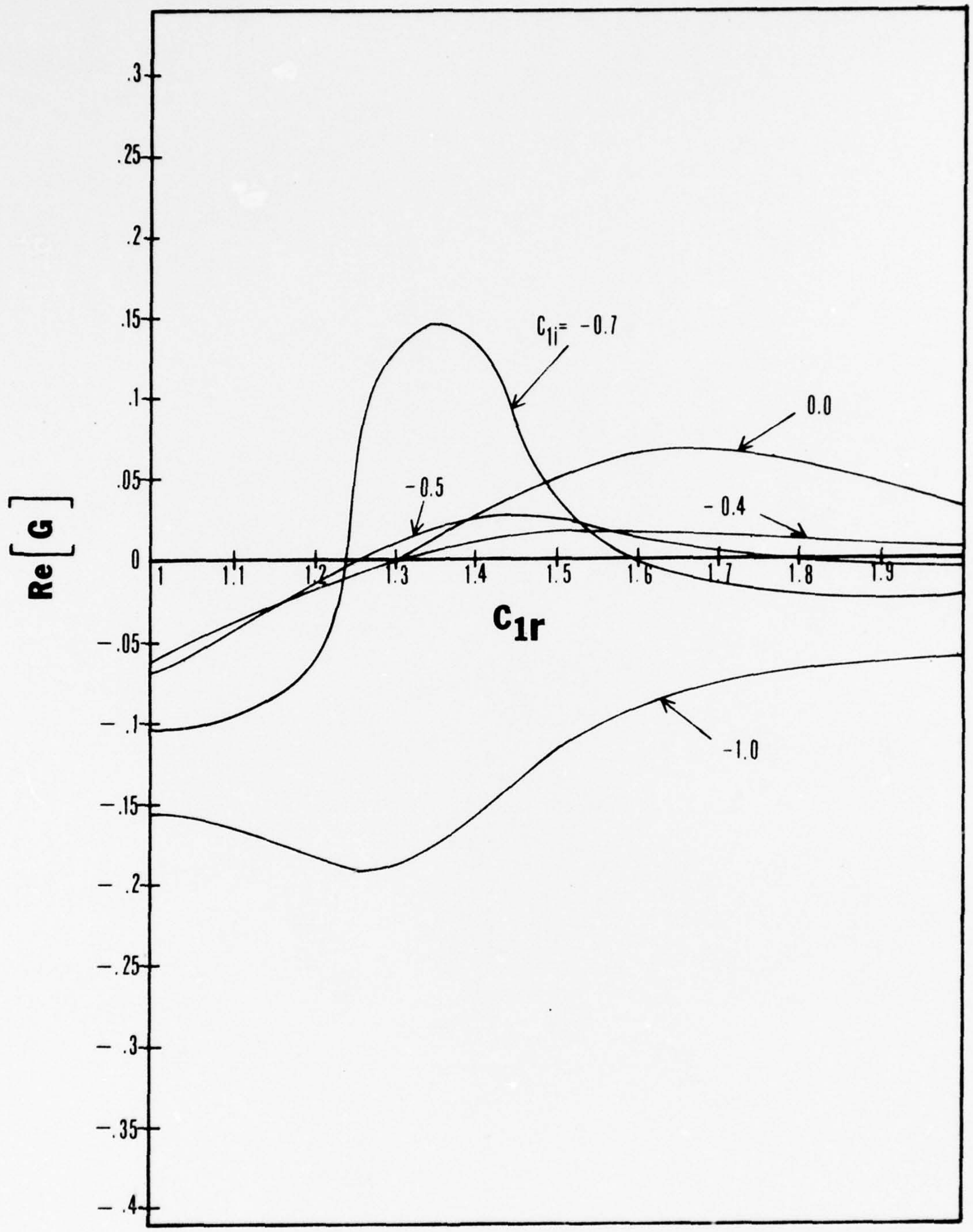


FIG. 5a. RE (G) VS RE (CI) WITH IM (CI) AS A PARAMETER

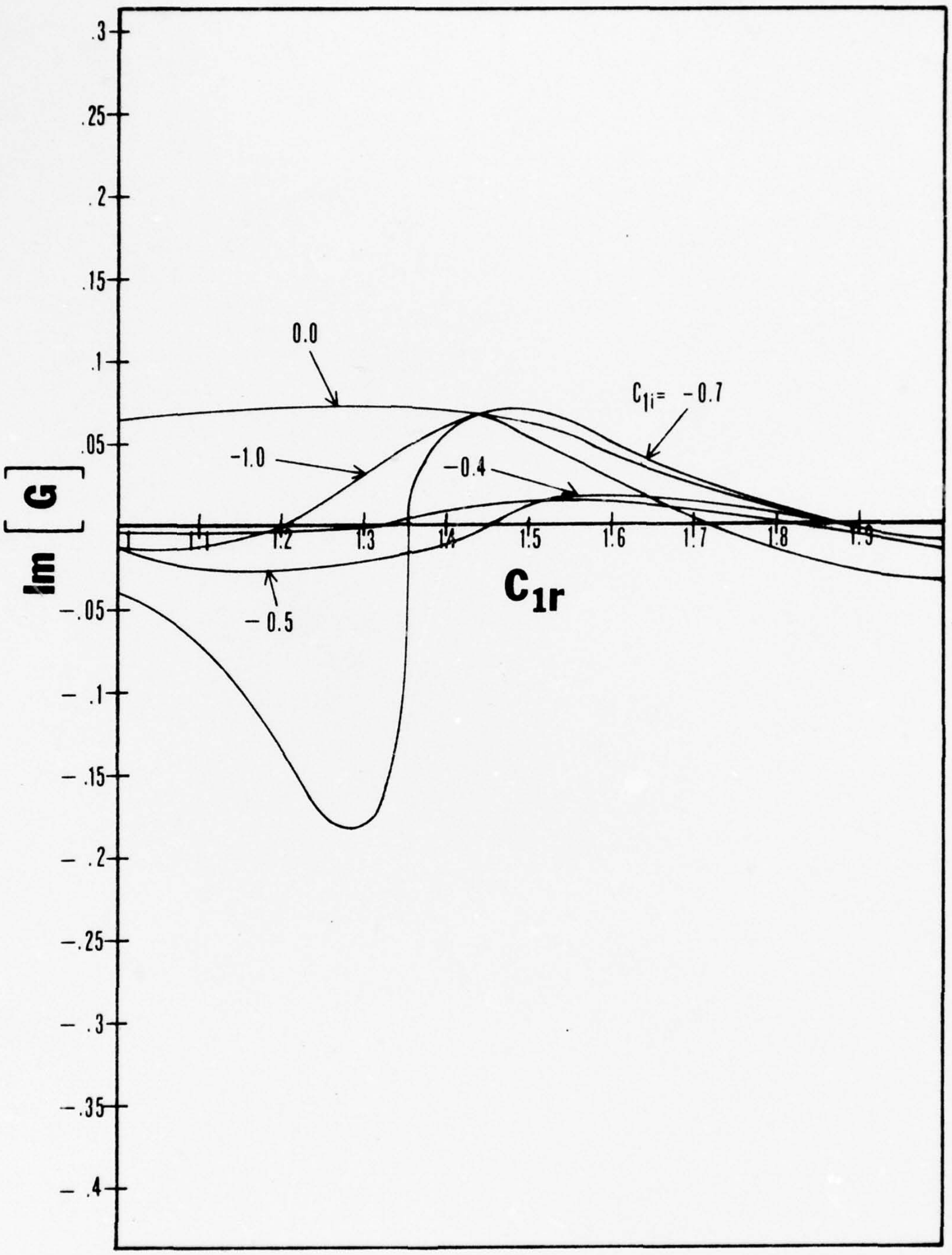


FIG. 5b. IM (G) VS RE(C_1) WITH IM(C_1) AS A PARAMETER

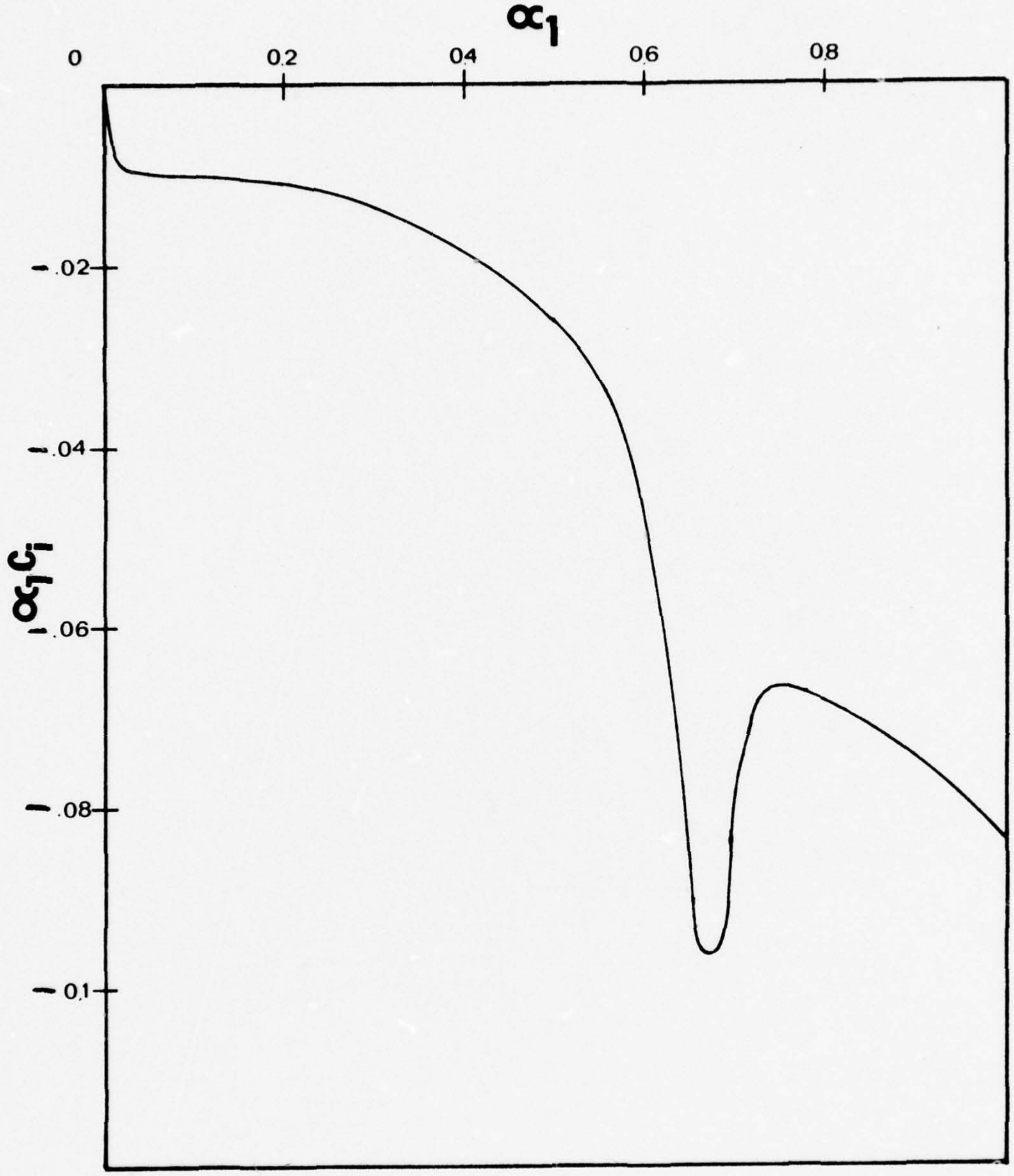


FIG. 6 a. AMPLIFICATION CURVE FOR EIGENVALUE C_{11}

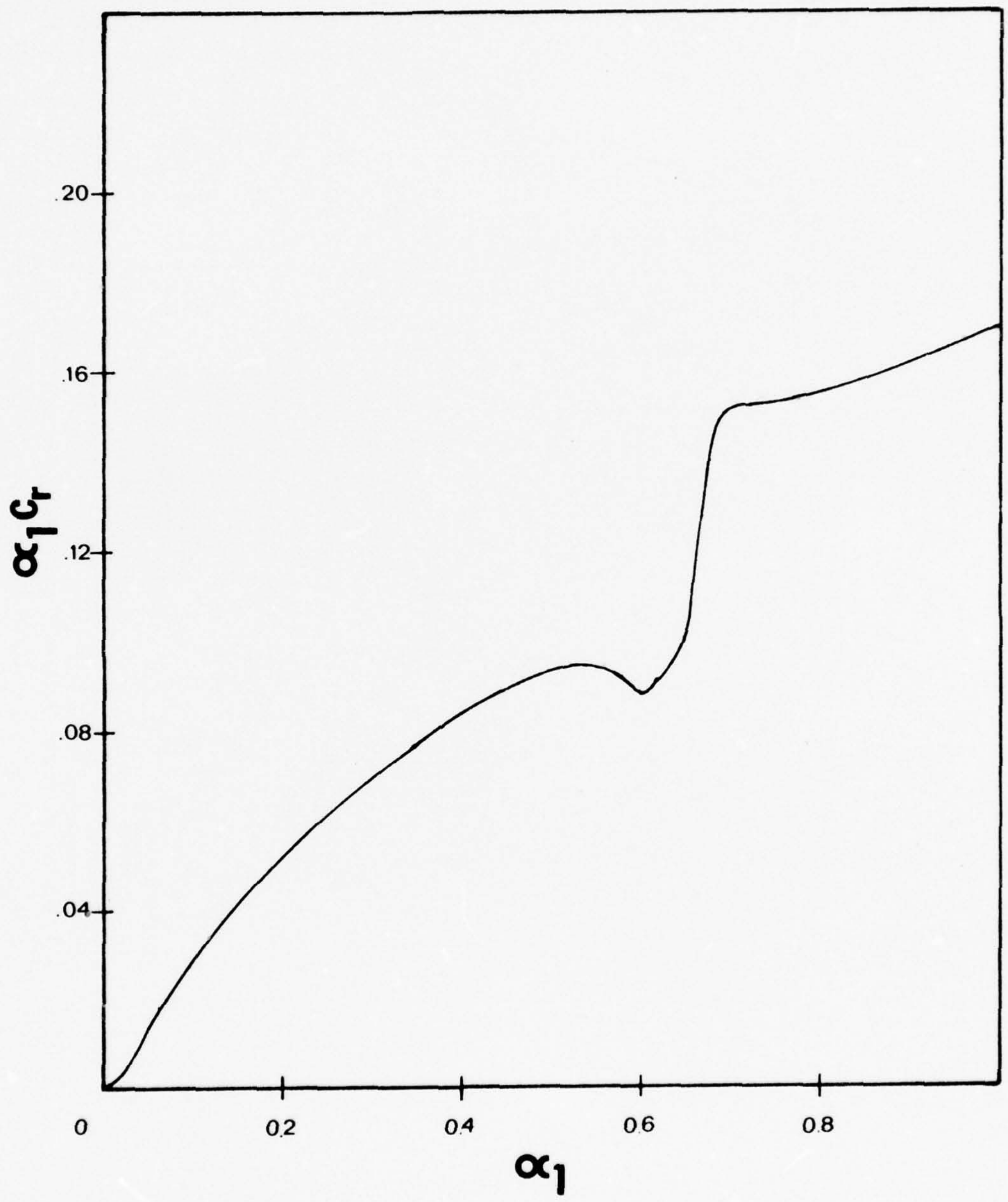


FIG. 6 b. PHASE VELOCITY CURVE FOR EIGENVALUE C_{11}

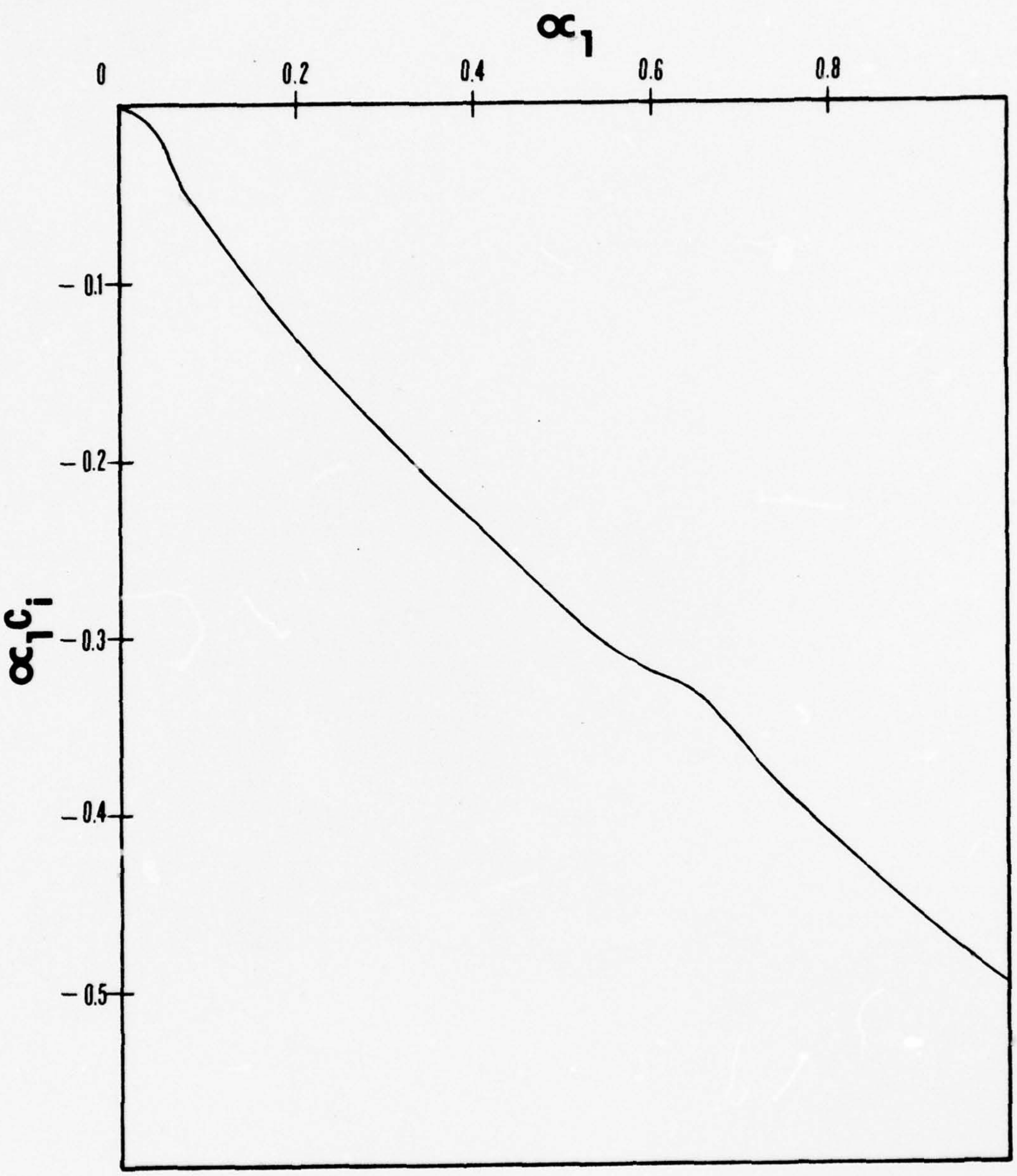


FIG. 7a AMPLIFICATION CURVE FOR EIGENVALUE C_{12}

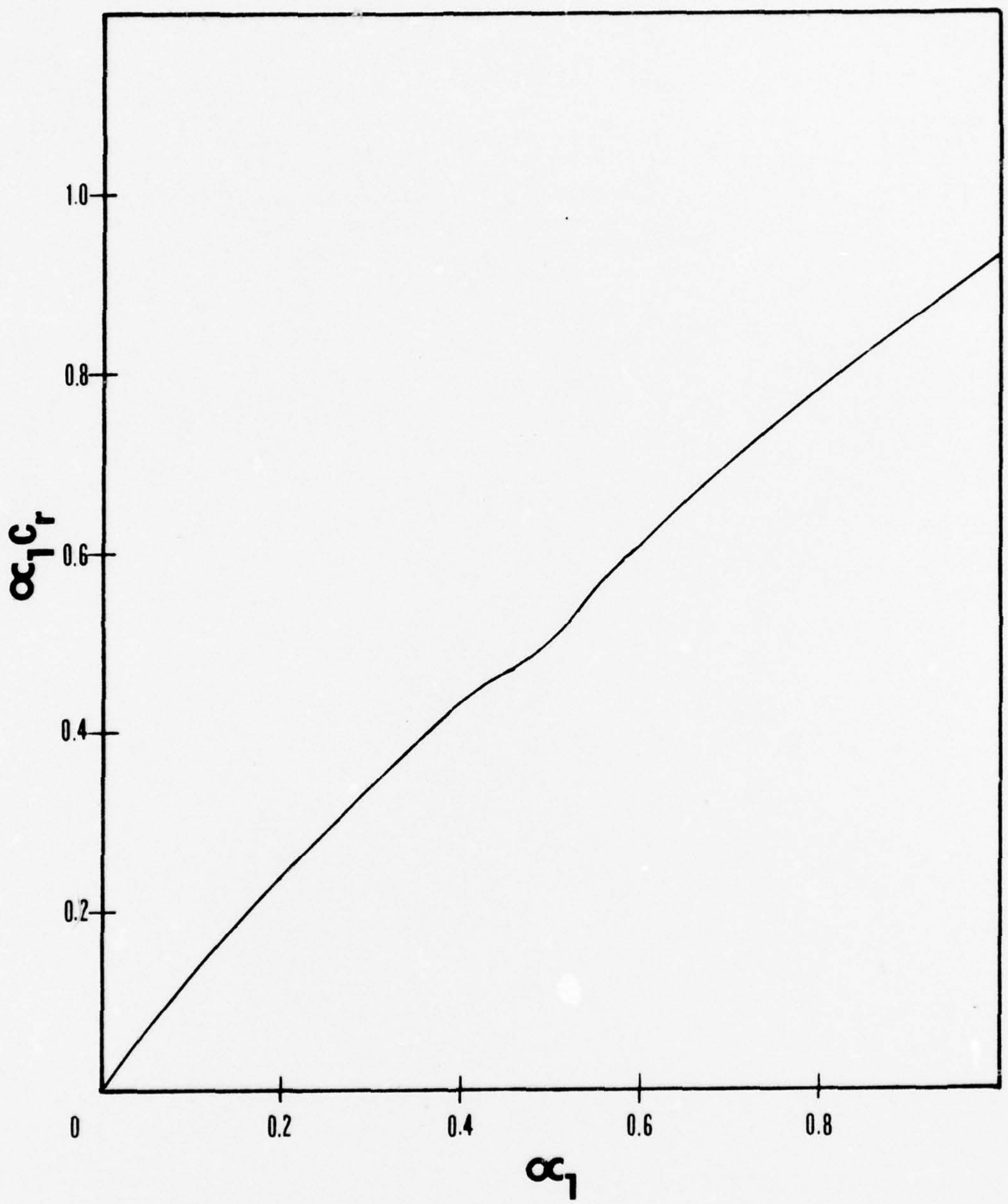


FIG. 7 b. PHASE VELOCITY CURVE FOR EIGENVALUE C_{12}

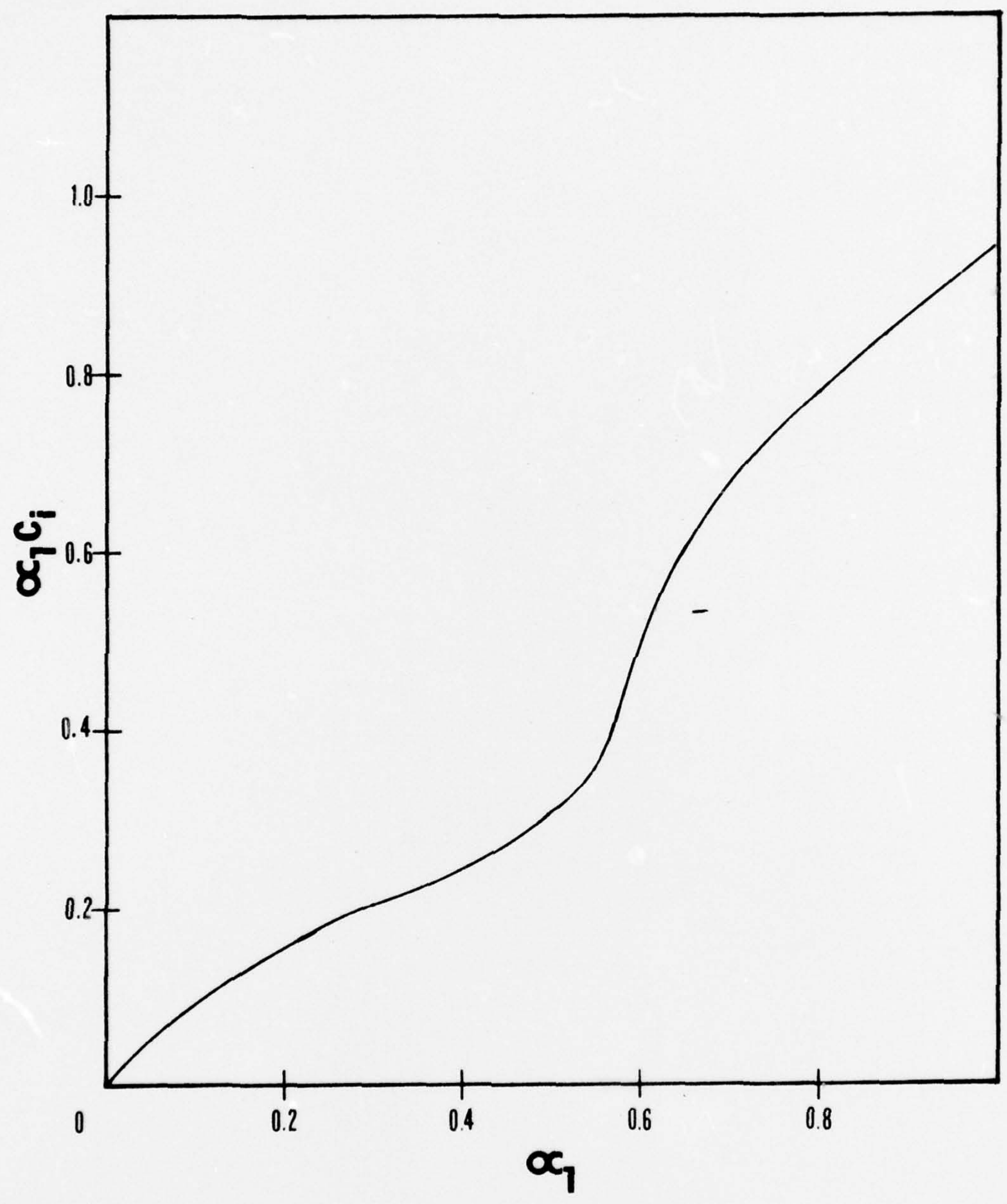


FIG. 8 a. AMPLIFICATION CURVE FOR EIGENVALUE C_{13}

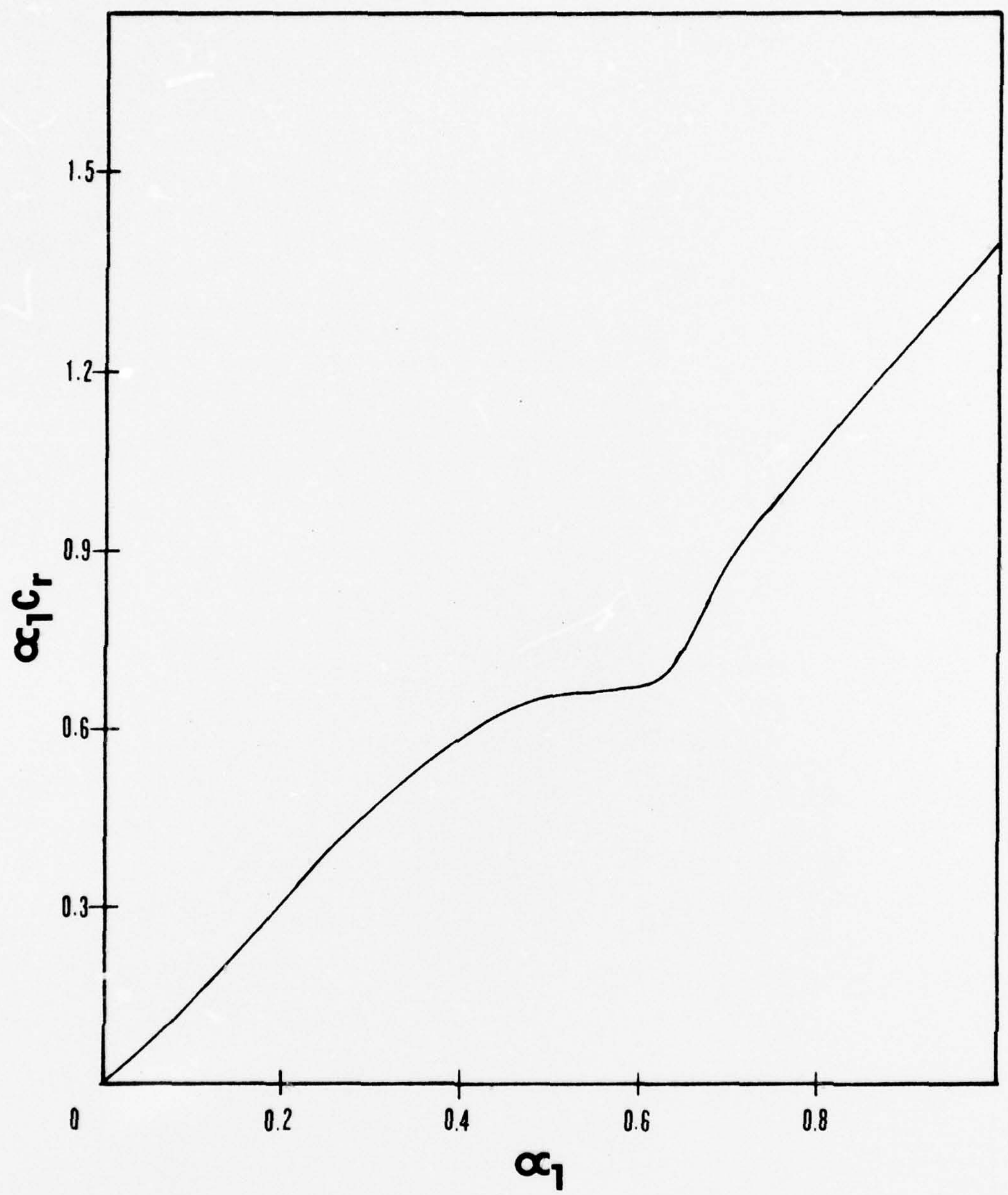


FIG. 8 b. PHASE VELOCITY CURVE FOR EIGENVALUE C_{13}

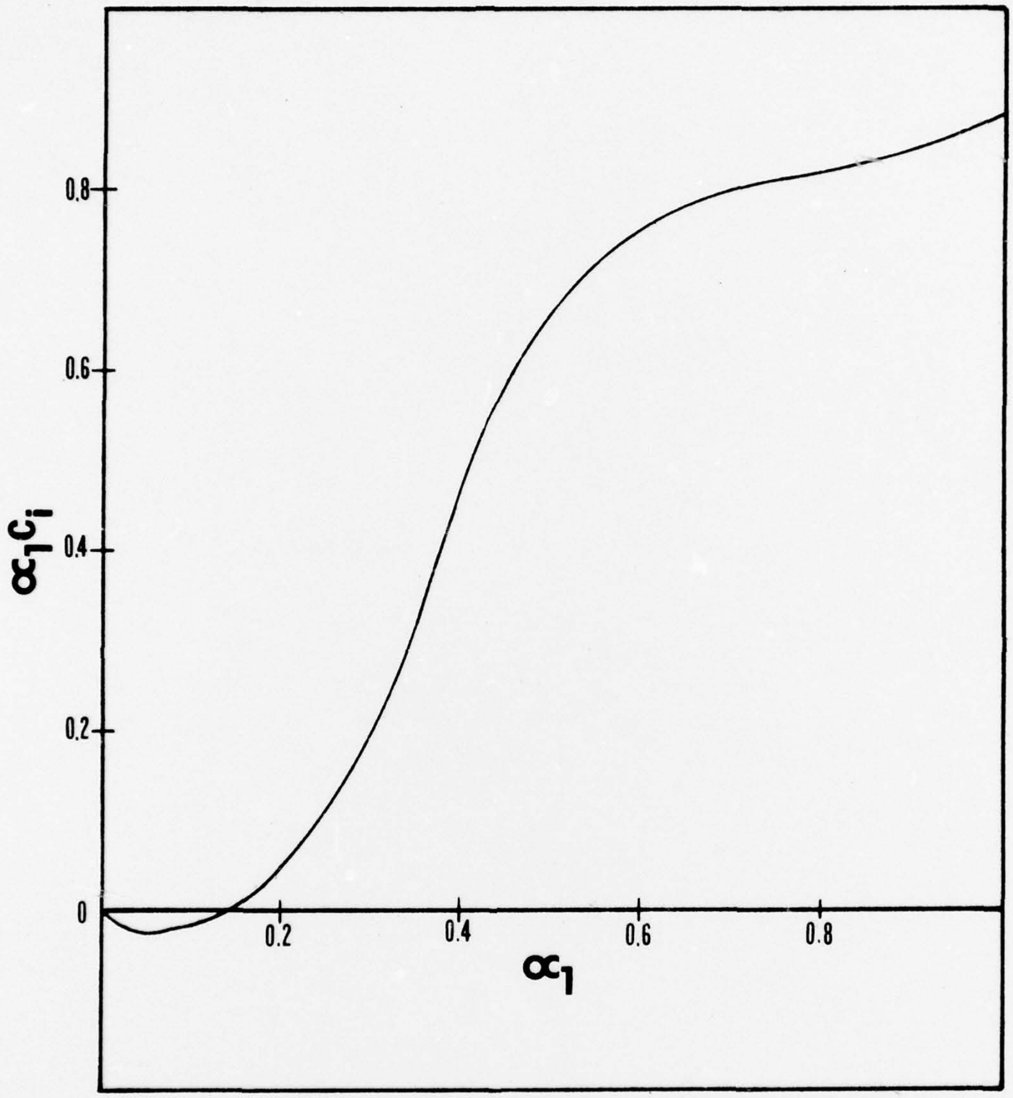


FIG. 9a. AMPLIFICATION CURVE FOR EIGENVALUE C_{14}

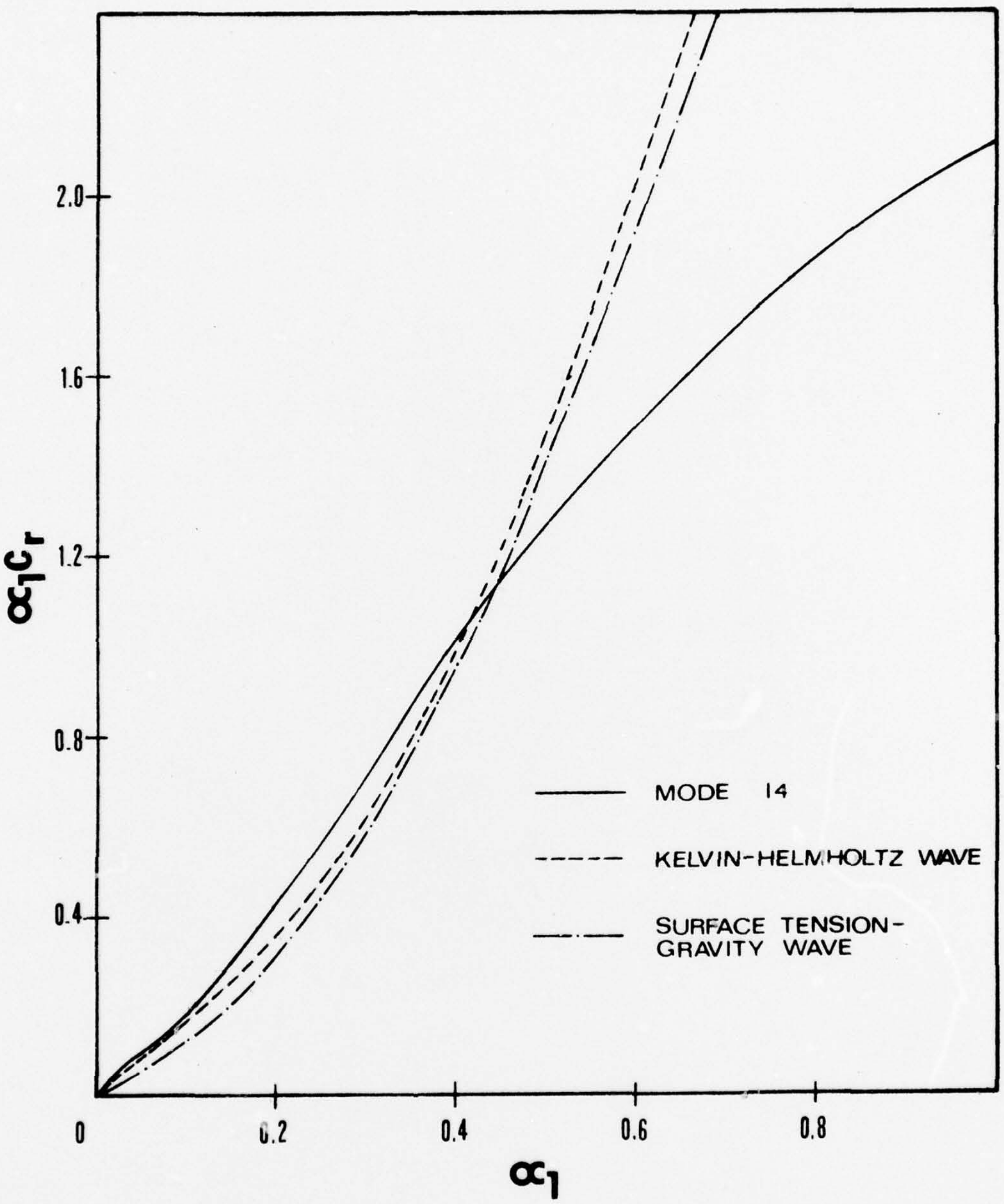


FIG. 9 b. PHASE VELOCITY CURVE FOR EIGENVALUE C_{14}; SURFACE TENSION-GRAVITY WAVES AND KELVIN-HELMHOLTZ WAVES

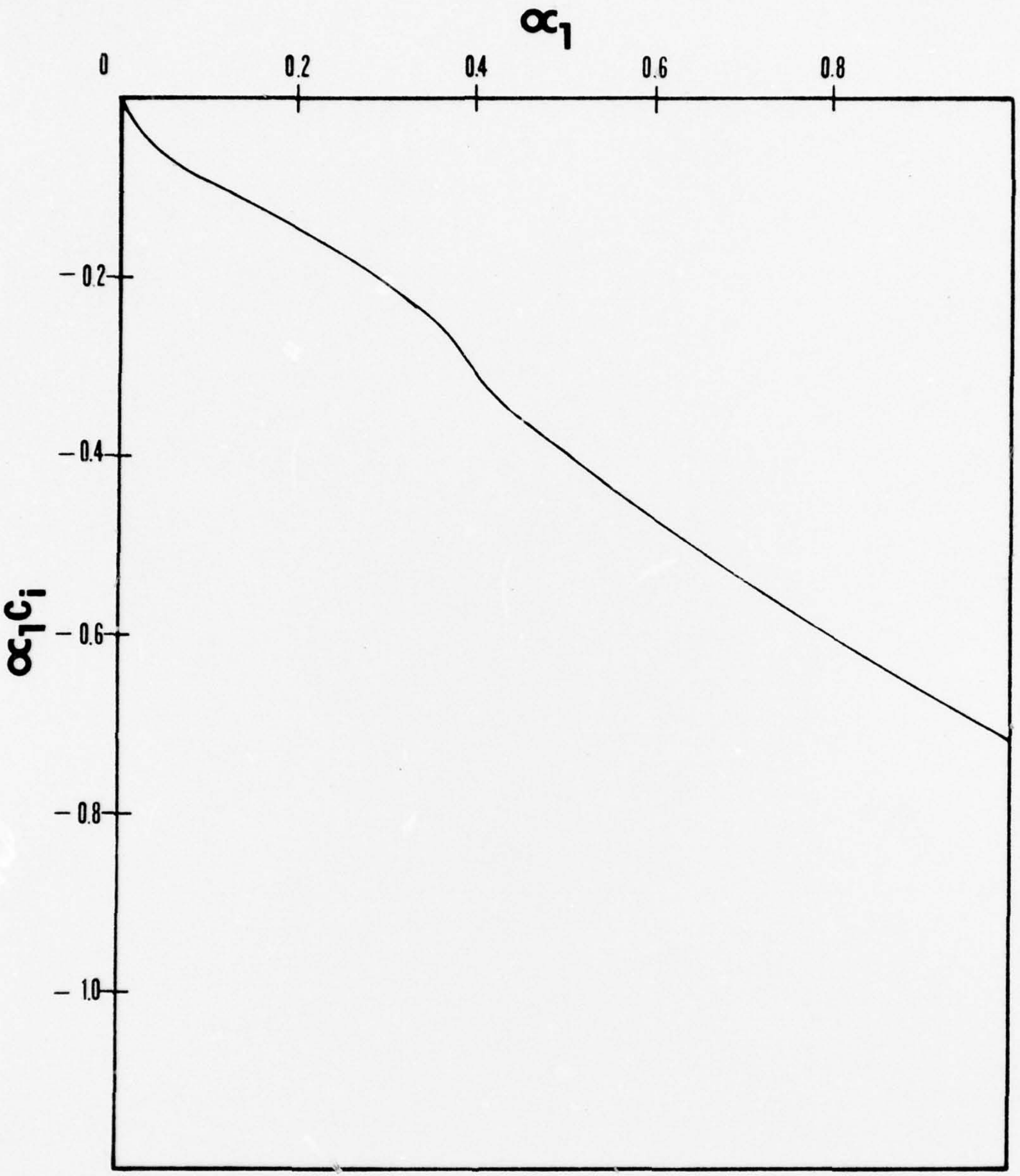


FIG. 10 a. AMPLIFICATION CURVE FOR EIGENVALUE C_{15}

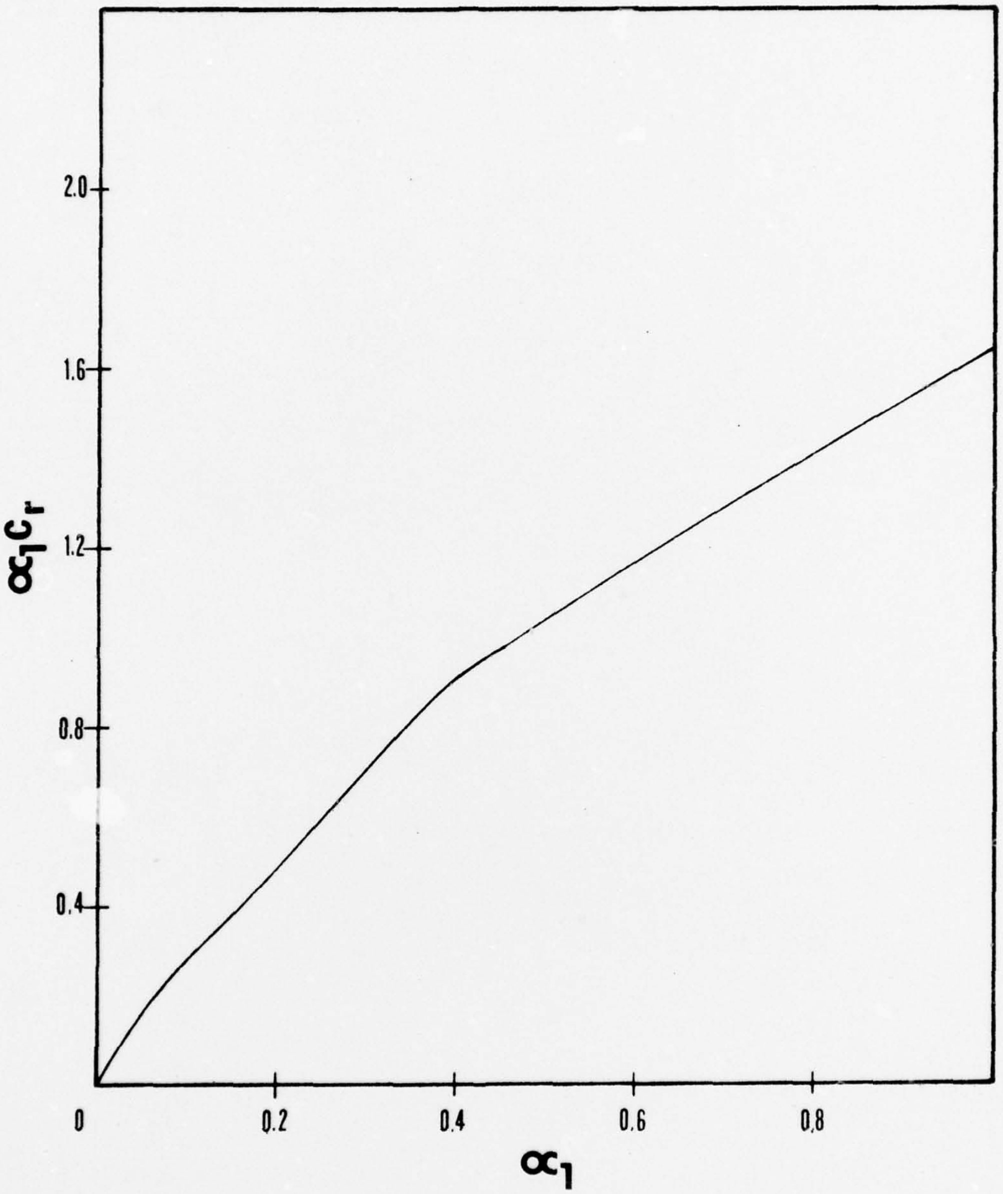


FIG. 10 b. PHASE VELOCITY CURVE FOR EIGENVALUE C_{15}

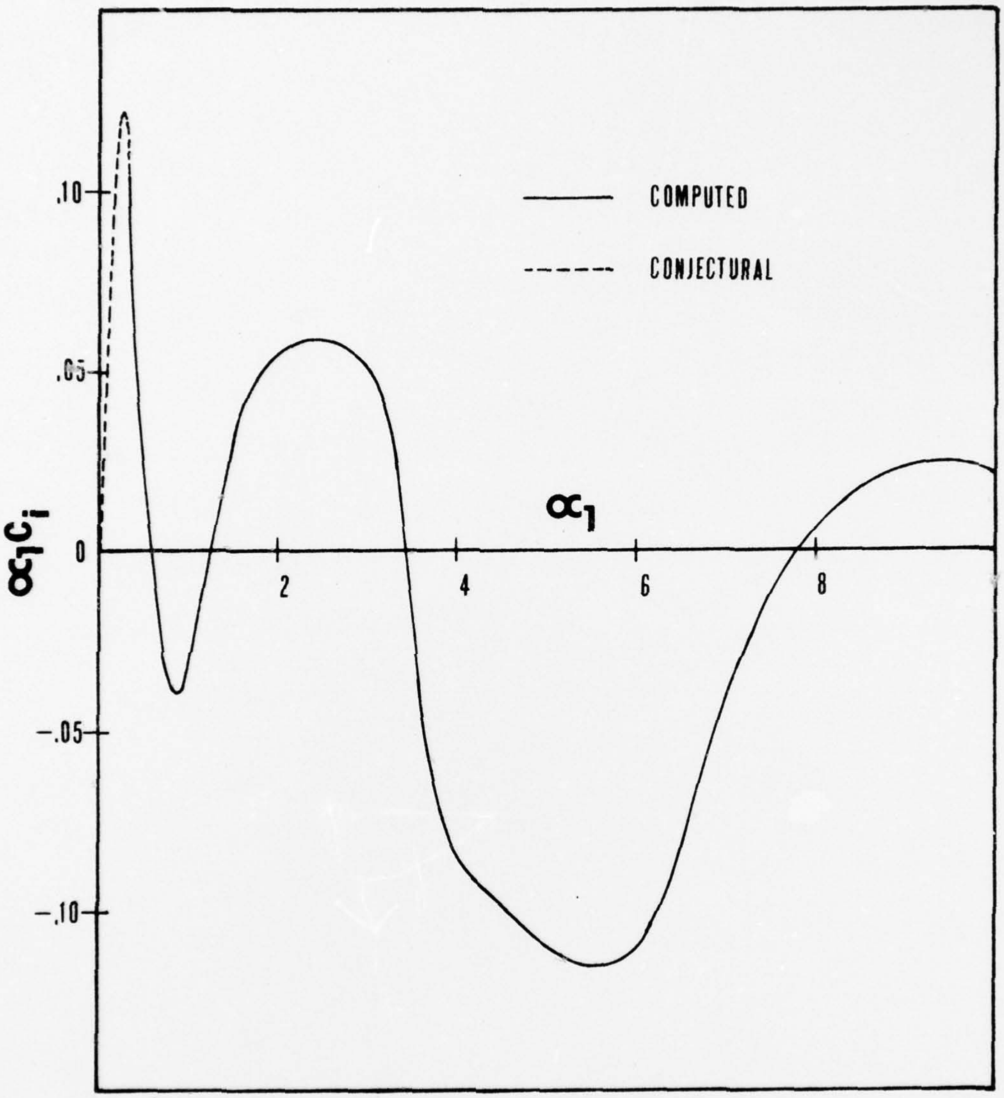


FIG. 11 a. AMPLIFICATION CURVE FOR EIGENVALUE C_{16}

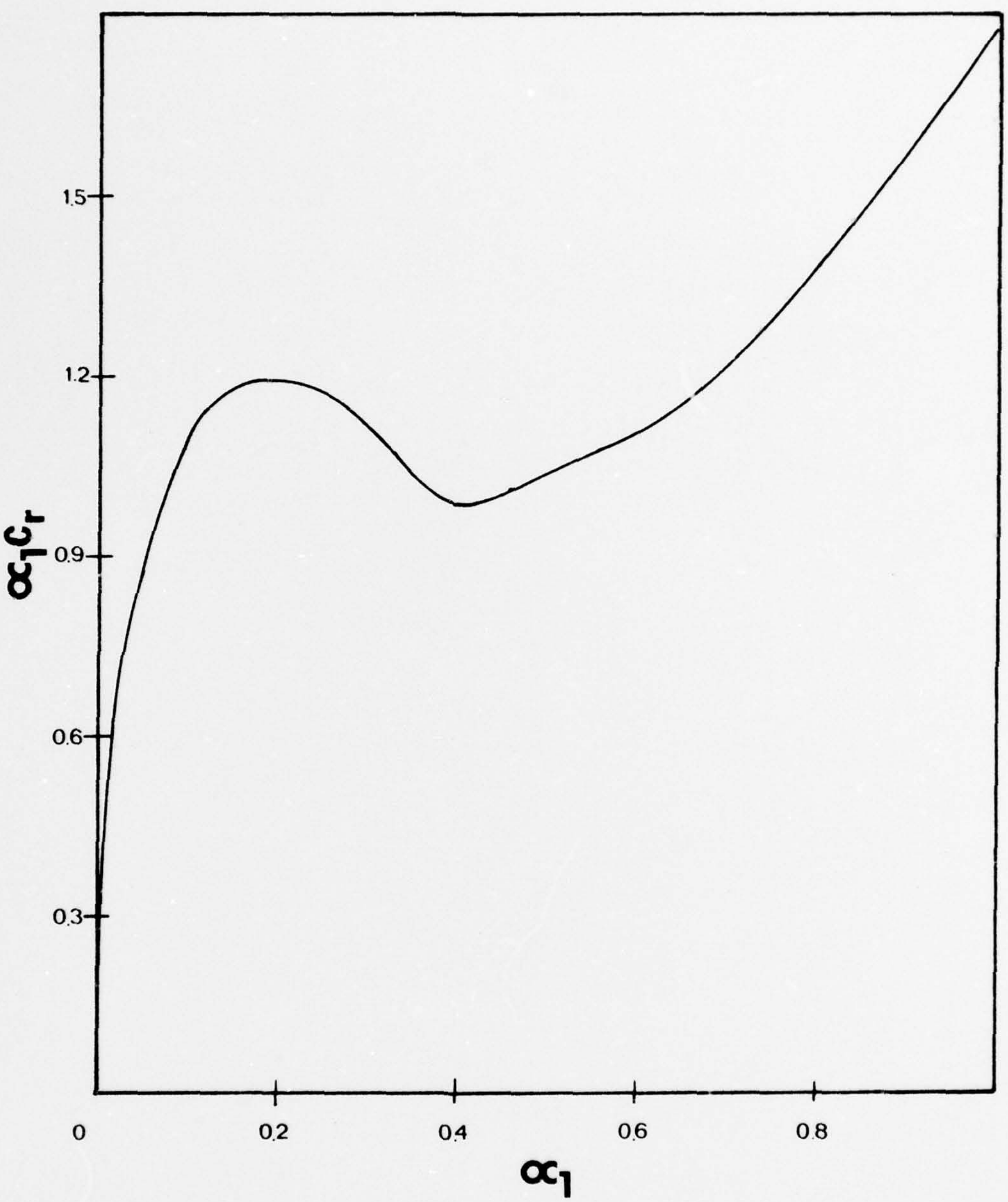


FIG. 11 b. PHASE VELOCITY CURVE FOR EIGENVALUE C_{16}

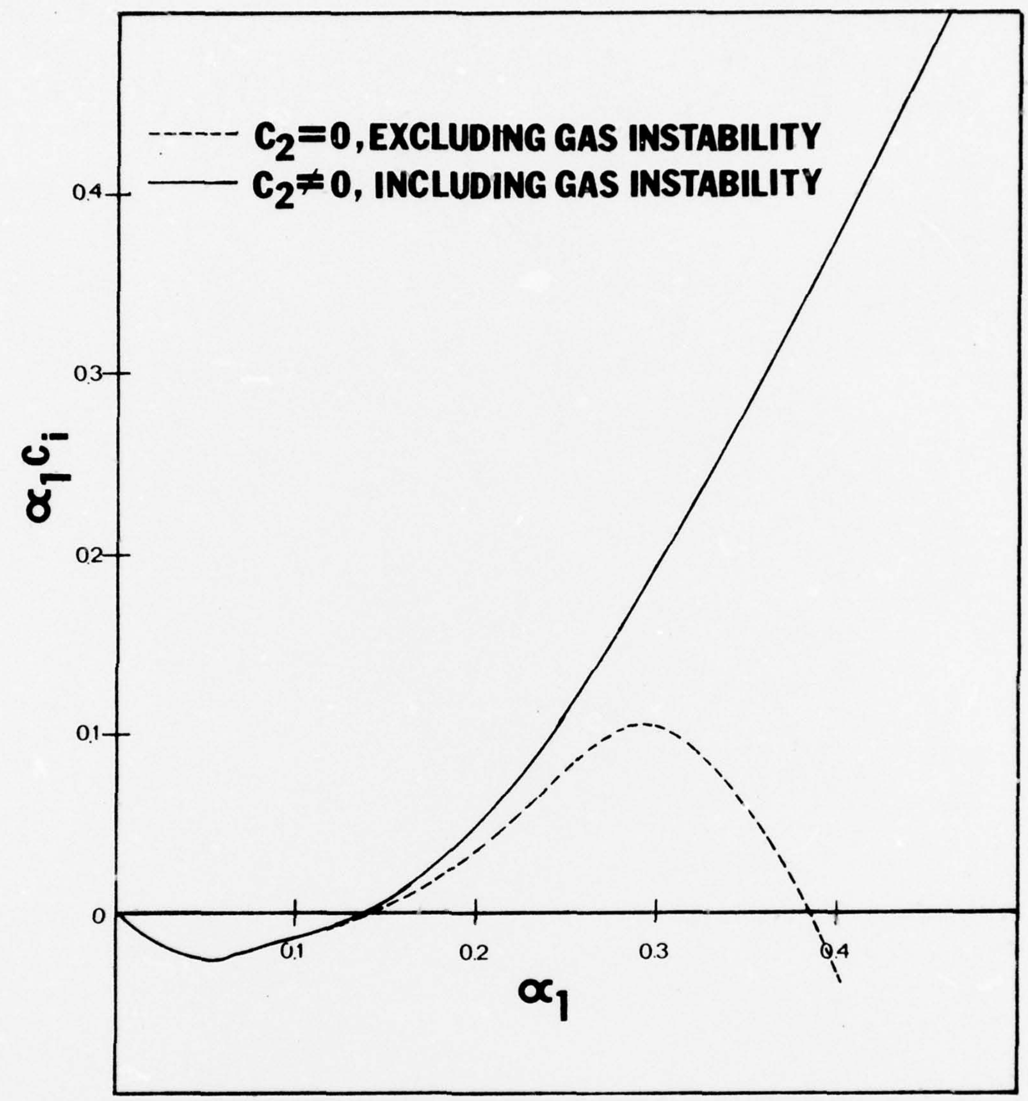


FIG. 12 a. COMPARISON OF AMPLIFICATION CURVES INCLUDING AND EXCLUDING INSTABILITIES IN GAS MOTION

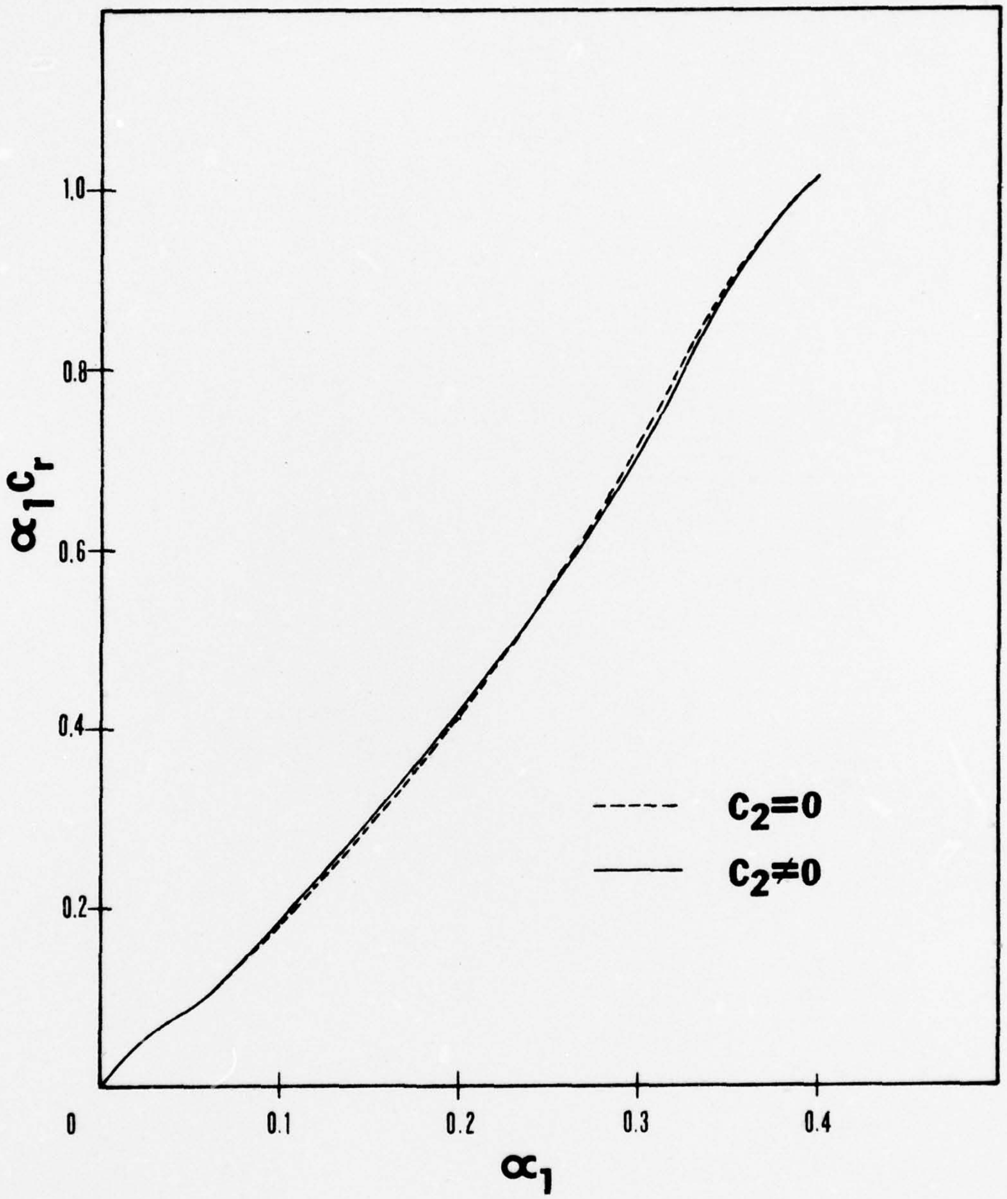


FIG. 12 b. COMPARISON OF PHASE VELOCITY CURVES INCLUDING AND EXCLUDING INSTABILITIES IN GAS MOTION

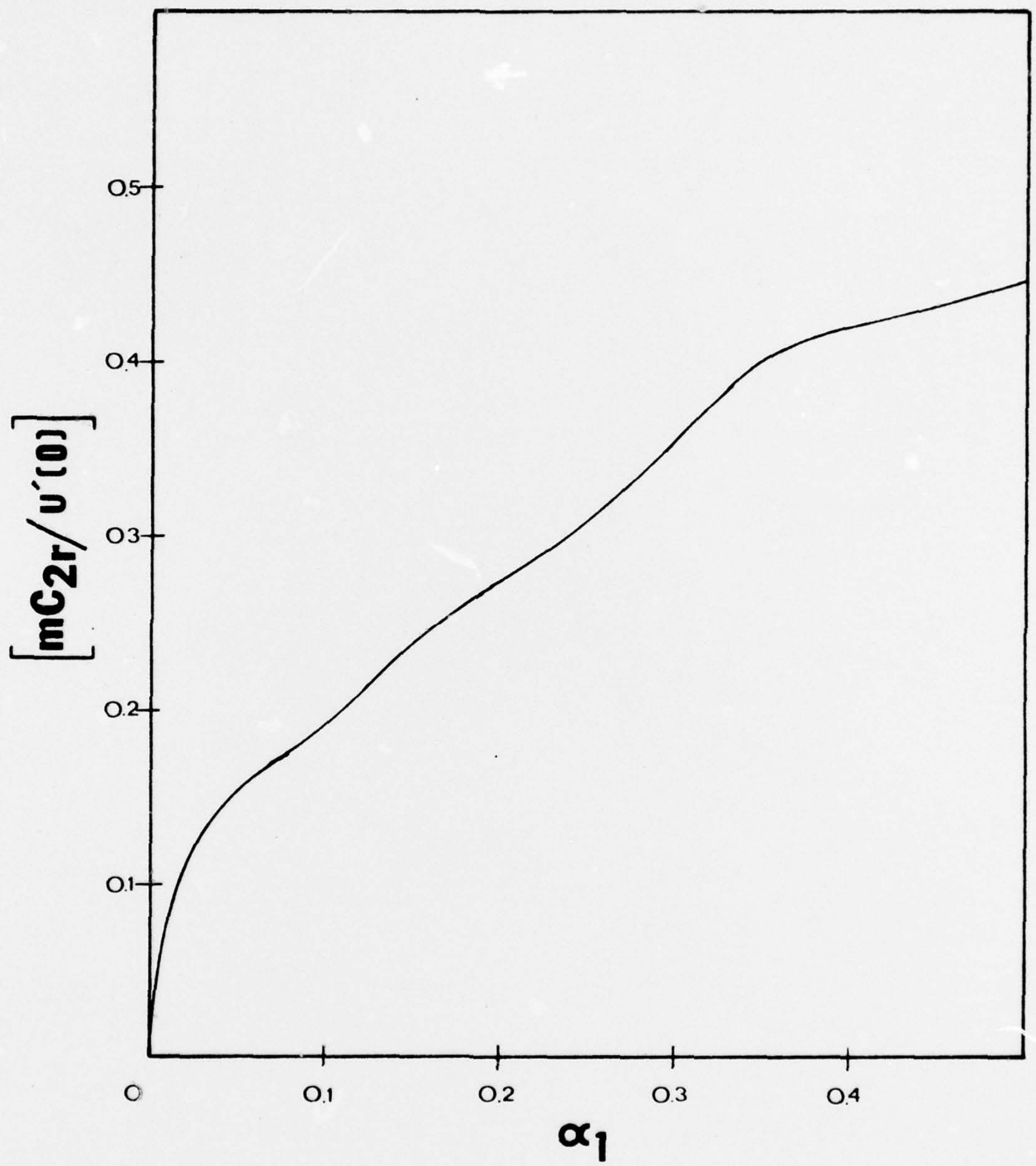


FIG. 13 VARIATION OF BENJAMIN'S PARAMETER WITH WAVE NUMBER

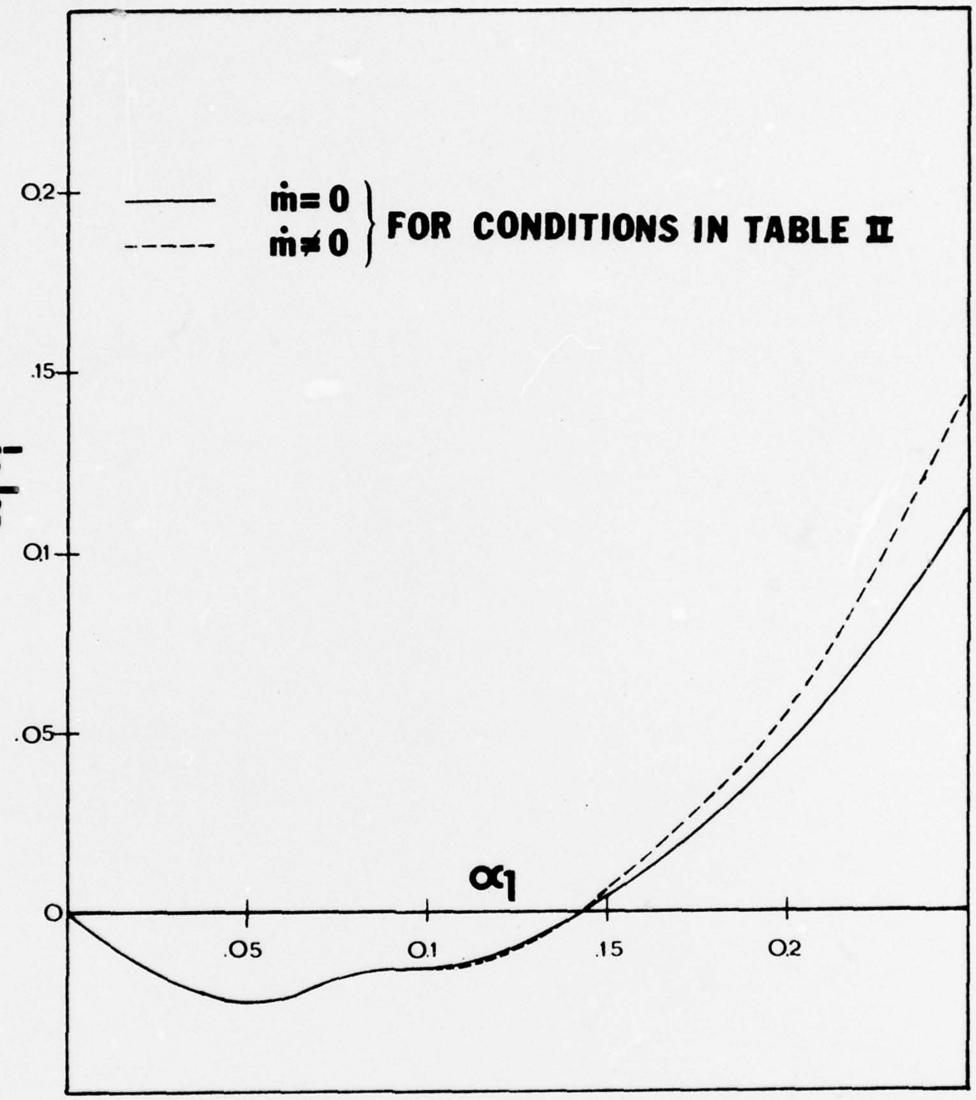


FIG. 14 a. EFFECT OF MASS TRANSFER ON MODIFIED KELVIN-HELMHOLTZ MODE (amplification curve)

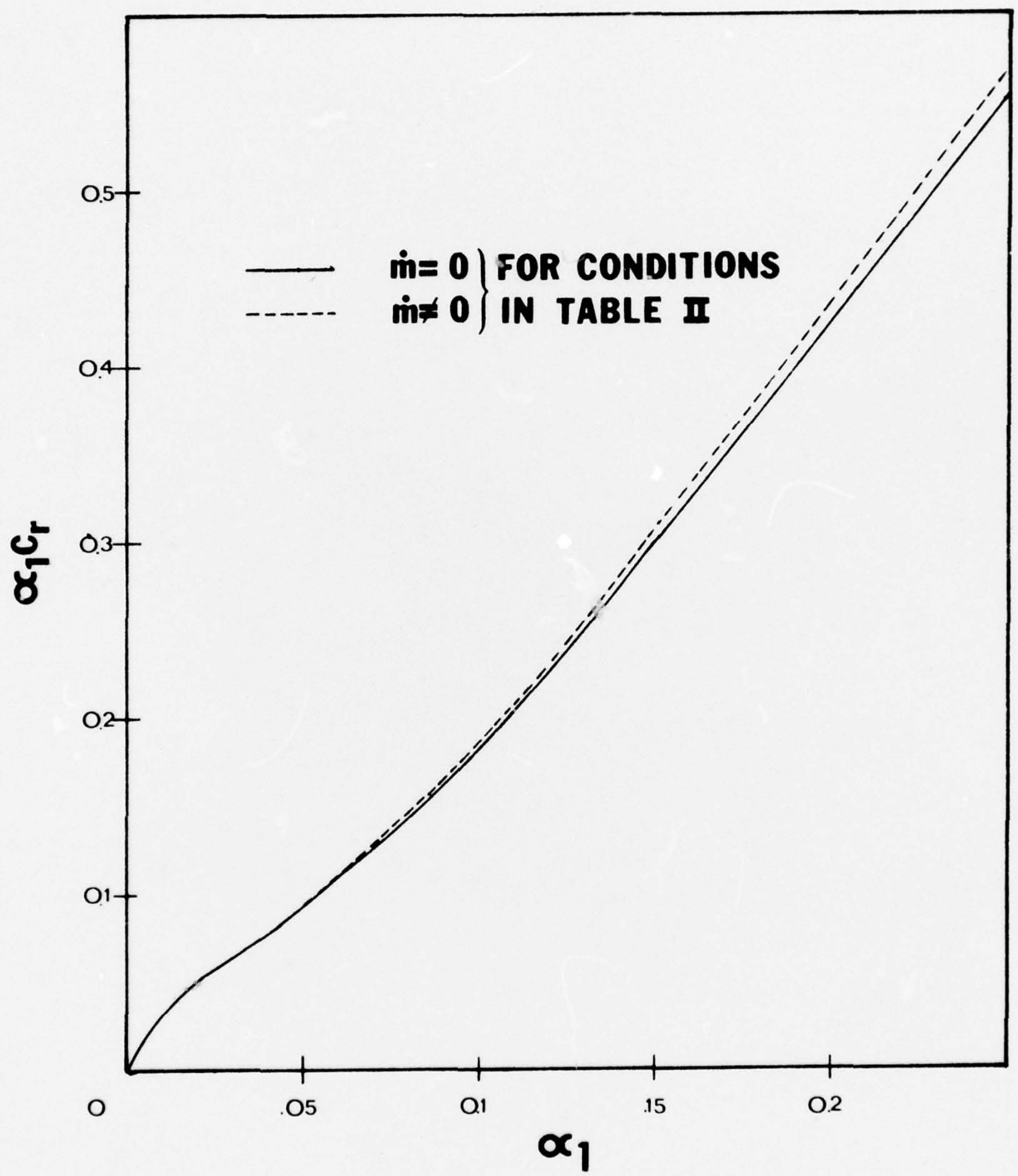


FIG. 14 b. EFFECT OF MASS TRANSFER ON MODIFIED KELVIN-HELMHOLTZ MODE (phase velocity curve)